NASA TN D-196

NASA TN D-196

7N-34

197728

TECHNICAL NOTE

D-196

ANALYSIS OF TEMPERATURE DISTRIBUTION AND RADIANT HEAT TRANSFER ALONG A RECTANGULAR FIN OF CONSTANT THICKNESS

By Seymour Lieblein

Lewis Research Center
Cleveland, Ohio

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

WASHINGTON

MAR 1 . 1960

ERRATA

NASA Technical Note D-196

By Seymour Lieblein

November 1959

"Units of length" should be deleted from the definitions of L and X (pp. 3 and 4); from the abscissas of figures 5(a) (p. 42), 5(b) (p. 43), 5(c) (p. 44), 8 (p. 47), 9 (p. 48), 10 (p. 49), 14(a) (p. 53), 14(b) (p. 54), 14(c) (p. 55), 15 (p. 56), 16 (p. 57); and from the key of figure 11 (p. 50).

Page 8, equation (7): The quantity T^4 should be T^5.

The unit "$ft/^{o}R^{3/2}$" should be changed to "$^{o}R^{-3/2}$" in the abscissa of figures 3(a) (p. 38), 3(b) (p. 39), 3(c) (p. 40), and 4 (p. 41); and in the abscissa and key of figure 7 (p. 46).

NASA - Langley Field, Va.

Issued 3-14-60

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

TECHNICAL NOTE D-196

ANALYSIS OF TEMPERATURE DISTRIBUTION AND RADIANT HEAT TRANSFER ALONG A RECTANGULAR FIN OF CONSTANT THICKNESS

By Seymour Lieblein

SUMMARY

A theoretical analysis has been conducted of the one-dimensional steady-state radiant heat-transfer characteristics of rectangular fin plates of constant thickness with uniform heat source along the leading edge. The analysis was made for plates of both finite and infinite length with constant thermal properties and no convection. Results are presented in terms of the variation of the ratio of surface temperature to source temperature and of radiating effectiveness with generalized distance from the heat source for a wide range of ratios of environment sink-to-source temperatures.

Results show that environment sink temperature has a pronounced effect on the temperature profiles but only little effect on the variation of radiating effectiveness. Negligible influence on the total heat radiated is also observed up to ratios of sink-to-source temperature of about 0.4. For a given sink temperature ratio, the decrease in plate temperature and radiating effectiveness with distance from the heat source depends primarily upon the magnitude of the source temperature and, to a lesser extent, on the plate thickness, emissivity, and thermal conductivity.

The solutions are made applicable for a variety of environmental conditions through the determination of an equivalent sink temperature for the environment. Examples of special environments that can be treated in this manner and relations for determining the equivalent sink temperature for the environment are given. Design parameters for minimizing plate volume are derived, and applications of the results to practical radiator considerations such as weight optimizations and thermal stresses are indicated.

INTRODUCTION

In the absence of a heat-absorbing atmosphere in outer space, many space vehicles are faced with the problem of the dissipation of waste heat. Although some excess heat may be removed through absorption in the fuel in special cases, the prevailing means of heat dissipation will undoubtedly be by radiation. Preliminary analyses of space-vehicle powerplants (e.g., ref. 1) indicate that sizable surface areas are required to radiate the waste heat. Radiating surfaces may thus comprise a considerable part of the total weight. A knowledge of the characteristics of heat-radiating surfaces in space is therefore necessary for the evaluation and design of space-vehicle systems.

It is currently envisioned that heat-radiating surfaces in spacecraft will either be part of the vehicle skin, as in the case of earth-launched vehicles, or be separate component structures, as may be found in an orbital-launched system. For these systems, the radiating surfaces may either be continuous surfaces (in which the waste heat to be radiated is introduced either continuously or periodically along the surface) or may be composed directly of the tubes or channels carrying the heat-transfer fluid, or some combination of the two. In many instances, therefore, radiators will contain surfaces that will be required to both radiate and conduct heat along the surface. Several examples of such radiator configurations are shown schematically in figure 1. Furthermore, these radiators will be exposed to other heat-emitting surfaces such as the sun, a planet, or other structures. Thus, fundamentally, such radiators correspond to the problem of the combined effects of conduction and radiation on the net heat transfer from a surface to an arbitrary environment in space.

For analysis purposes, the continuous radiating surface with periodic distribution of heat sources can be approximated by the flat plate of constant thickness with a heat source along the leading edge of the plate. If it is further specified that the heat source be constant along the leading edge, the problem is reduced to a one-dimensional one. Such a situation in reality would correspond to a radiator to which the waste heat is transferred by means of condensation of working-fluid vapor (vapor heat cycle).

In the analysis presented herein, generalized steady-state heat-transfer characteristics are determined for fin plates of both infinite and finite length and constant thermal properties. Calculations of the variation of surface temperature and heat-transfer effectiveness with distance from the heat source are made by numerical means for a wide range of ratios of sink-to-source temperatures. The solutions are made applicable for a variety of special environmental conditions through the use of an equivalent sink temperature. Design parameters for minimizing fin plate volume are determined, and application of results to practical radiator considerations is indicated.

A similar analysis of the heat-transfer characteristics of a fin plate uniformly heated along an edge previously reported in reference 2 was not received until the present calculations were nearly completed. The present report is believed to be a more complete and detailed treatment of the problem and carries the solutions to a wider range of fin end and effective environmental temperatures. In addition, data on the variations of temperature and heat radiated along the fin are included herein.

SYMBOLS

A	surface area
A'	projected surface area normal to incoming radiation
B	factor in equation for net heat transfer from fin plate (eq. (A24))
C	constant of integration
E	emissivity factor in equation for net heat transfer from fin plate (eq. (A24))
e_λ	radiant energy contained in wavelength increment $d\lambda$
F	view or configuration factor for radiation between two surfaces
F'	view factor for radiation between the illuminated portion of a surface and another surface
f	function
k	thermal conductivity
L	plate generalized length parameter, $l\sqrt{\frac{\sigma E(10^9)}{kt}\left(\frac{T_0}{1000}\right)^3}$, units of length
l	plate length
m,n	exponents
N	total number of discrete surfaces in fin environment
Q	integrated heat radiated per unit time
Q_c	heat conducted per unit time
$\overline{Q}$	integrated heat radiated per unit time with entire fin at source temperature

S heat flux from concentrated heat source (e.g., 429 (Btu/hr)(sq ft) for solar radiation in neighborhood of earth)

S' spacing between radiator tubes

T plate temperature, deg abs

T_s sink temperature of environment, deg abs

T_0 source temperature, deg abs

T_s^* equivalent sink temperature of environment, deg abs

$\overline{T}$ effective average temperature for fin, deg abs

t plate thickness

V plate volume

W plate width

X plate generalized distance parameter, $x\sqrt{\frac{\sigma E(10^9)}{kt}\left(\frac{T_0}{1000}\right)^3}$, units of length

x distance along plate

α absorptivity

Δ change in quantity

ϵ hemispherical emissivity

η radiating effectiveness

θ angle between incoming radiation and normal to receiving surface, deg

λ wavelength of radiant energy

ρ reflectivity

σ Stefan-Boltzmann constant (e.g., 0.173×10^{-8} (Btu/hr)(sq ft)($^{\circ}R^4$))

Subscripts:

b enclosed body

d discrete surface

E-529

e	earth
f	fin strip surface
$\bar{f}$	fin total surface
i,j	integers referring to discrete surfaces
l	terminal value at end of plate ($x = l$)
l'	with length l held fixed
min	minimum
S	concentrated heat source
s	sink environment
t	with thickness held fixed
W	with width held fixed
x	distance position along plate
λ	monochromatic value
0	heat source at leading edge of plate ($x = 0$)
1	"upper" surface of fin plate
2	"lower" surface of fin plate
∞	infinitely long plate

ANALYSIS

Assumptions

The analysis considers the general case of a flat fin plate of constant thickness t and length l uniformly heated along its leading edge at $x = 0$, as shown in figure 2. Heat is conducted along the plate from the source and is radiated from both surfaces to the surrounding environment. The expected variation of plate temperature from its value at the source T_0 to its terminal value T_l and the variation in net integrated radiant heat transfer along the plate are indicated in the lower part of the figure.

The specific assumptions used in the development of relations for the heat-transfer characteristics of the fin plate are as follows:

(1) The heat transfer is invariant with time (steady state).

(2) Heat is transferred out of the plate only by radiation (no convection or external conduction) through a nonabsorbing medium.

(3) Thermal properties are constant.

(4) There is no conduction in the y-direction.

(5) The plate temperature is effectively constant across the thickness t at all x positions (valid for $t \ll W$ and $t \ll l$).

(6) The heat loss from the two exposed side edges is sufficiently small so that they can be considered to be thermally insulated (valid for $t \ll W$).

(7) The heat loss from the end edge, when exposed, is sufficiently small so that the end edge also can be considered to be thermally insulated (valid for $t \ll l$).

The problem is thus reduced to a mathematical formulation for one-dimensional steady-state heat transfer under combined radiation and conduction.

Governing Equation for Temperature Distribution

The fundamental differential equation for the temperature distribution in linear heat flow under combined radiation and conduction has been established in the literature (e.g., ref. 3, p. 135). For completeness, the derivation for the specific case of the rectangular fin plate will be developed herein.

The differential equation for the temperature variation is obtained from consideration of the heat balance for an elemental strip $W\,dx$ as shown in figure 2. The local heat conduction through the cross-sectional area tW at position x is given by Fourier's law as

$$Q_{c,x} = -k(tW)\left(\frac{dT}{dx}\right)_x \tag{1}$$

At $x + dx$, the heat conduction is

$$Q_{c,x+dx} = -k(tW)\left(\frac{dT}{dx}\right)_x + \frac{dQ_{c,x}}{dx}\,dx \tag{2}$$

E-529

E-529

The net internal heat conduction across the shaded element is then obtained as

$$dQ_c = -k(tW)\left(\frac{d^2T}{dx^2}\right) dx \tag{3}$$

In the absence of any convection or external conduction, the external heat loss from the strip is given by the difference between the heat radiated from both surfaces of the strip and the heat received by the strip surfaces by radiation from the external environment. In practical situations, the external environment of the fin may be composed of a large enclosing surface or several discrete surfaces in conjunction with one or more concentrated heat sources. In general, the determination of the net heat transfer between the fin and an arbitrary environment is extremely difficult, since the solution of complex integral equations is required (e.g., ref. 4, pp. 26-47). However, simple approximations for the net heat transfer have been obtained in special cases through the use of simplifying assumptions. Several practical situations for which simple relations are possible are discussed in appendix A.

For the special cases considered therein, the net heat radiated from the surfaces of the elemental strip can be expressed in the general form

$$dQ = \sigma E\left(T^4 - T_s^{*4}\right)W\ dx \tag{4}$$

where E is a factor involving the fin surface emissivities and T_s^* is an equivalent sink temperature for the particular environment, as defined in appendix A. The specific conditions required to make equation (4) an acceptable representation of the net heat transfer are also given in appendix A. For example, for the elementary case of a fin plate with equal thermal properties on both surfaces orbiting around the earth (temperature T_e) with one surface always parallel to the earth,

$$E = 2\epsilon \tag{5a}$$

and

$$T_s^{*4} = \frac{1}{2}\left(\frac{\alpha}{\epsilon}\right)T_e^4 + \left(\frac{\alpha}{\epsilon}\right)\frac{S\cos\theta}{2\sigma} \tag{5b}$$

where S is the solar constant and θ is the angle between the sun's radiation and the normal to the plate. It is also shown in appendix A that, for the environmental conditions considered, the factor E and the equivalent sink temperature T_s^* are both effectively independent of x.

For steady state, the net heat loss from the surface (eq. (4)) is equal to the change in the internal heat conducted across the strip given by equation (3). Thus,

$$W\left(\frac{d^2T}{dx^2}\right)dx = \frac{\sigma E}{kt}\left(T^4 - T_s^{*4}\right)W\ dx$$

or

$$\frac{d^2T}{dx^2} = \frac{\sigma E}{kt}\left(T^4 - T_s^{*4}\right) \tag{6}$$

Equation (6) is readily integrated once through the use of the substitution $p = dT/dx$ to give (since the temperature slope is everywhere negative)

E-529

$$\frac{dT}{dx} = -\sqrt{\frac{2\sigma E}{5kt}\left(T^4 - 5T_s^{*4}T\right) + C} \tag{7}$$

The constant of integration C is evaluated from the boundary condition at $x = l$. If the plate is of infinite length, it is clear that at $x = l = \infty$ the temperature slope must approach zero, so that $(dT/dx)_l = 0$. For this condition, from equation (7),

$$C_\infty = -\frac{2\sigma E}{5kt}\left(T_l^5 - 5T_l T_s^{*4}\right) \tag{8}$$

If the plate is of finite length and one of a series of connected plates with repeated temperature variations (continuous surface with intermittent heat sources), then $x = l$ represents the point of minimum temperature between the sources. Here again, the boundary condition is identically $(dT/dx)_l = 0$, and equation (8) holds. If the plate is of finite length and the end edge is exposed to an environment, a net radiant heat loss will occur from the end surface area Wt. However, if $t \ll l$, as shown in appendix B, this edge loss can be neglected, and, again, equation (8) will be satisfactory.

With the constant of integration given by equation (8), the temperature gradient, from equation (7), is

$$\frac{dT}{dx} = -\sqrt{\frac{2\sigma E}{5kt}}\sqrt{\left(T^5 - T_l^5\right) - 5T_s^{*4}(T - T_l)} \tag{9}$$

If all temperatures are expressed as a ratio of the source temperature, equation (9) becomes

$$\frac{d(T/T_0)}{dx} = -0.6325\sqrt{\frac{\sigma E(10^9)}{kt}\left(\frac{T_0}{1000}\right)^3}\sqrt{\left[\left(\frac{T}{T_0}\right)^5 - \left(\frac{T_l}{T_0}\right)^5\right] - 5\left(\frac{T_s^*}{T_0}\right)^4\left(\frac{T}{T_0} - \frac{T_l}{T_0}\right)} \tag{10}$$

Furthermore, since the terms in the first radical are all independent of x and nondimensional, the independent distance variable can be expressed for simplicity as the generalized distance parameter

$$X \equiv x\sqrt{\frac{\sigma E(10^9)}{kt}\left(\frac{T_0}{1000}\right)^3} \tag{11}$$

so that the basic differential equation for the temperature variation along the plate becomes

$$\frac{d(T/T_0)}{dX} = -0.6325\sqrt{\left[\left(\frac{T}{T_0}\right)^5 - \left(\frac{T_l}{T_0}\right)^5\right] - 5\left(\frac{T_s^*}{T_0}\right)^4\left(\frac{T}{T_0} - \frac{T_l}{T_0}\right)} \tag{12}$$

Thus, the equation is reduced to a form involving only the three independent variables X, T_l/T_0, and T_s^*/T_0.

Solutions for the temperature-ratio variation along the plate were made for the three basic situations of the infinite plate with zero sink temperature, the infinite plate with a range of sink-to-source temperatures, and the finite plate with a range of sink-to-source temperatures. For the plate of infinite length, the boundary condition requires that no net heat be radiated at infinity, and, therefore, $T_l = T_s^*$ and $(dT/dx)_l = 0$. In this case, equation (12) reduces to

$$\frac{d(T/T_0)}{dX} = -0.6325\sqrt{\left(\frac{T}{T_0}\right)^5 - 5\left(\frac{T_s^*}{T_0}\right)^4\left(\frac{T}{T_0}\right) + 4\left(\frac{T_s^*}{T_0}\right)^5} \tag{13}$$

For the special case of the infinite plate with zero equivalent sink temperature, equation (13) reduces further to a simple form that can be integrated directly to give

$$\frac{T}{T_0} = \left[\frac{1}{1 + 0.9487\ x\sqrt{\frac{\sigma E(10^9)}{kt}\left(\frac{T_0}{1000}\right)^3}}\right]^{2/3} \tag{14}$$

Solutions for the cases with nonzero sink temperatures (eqs. (12) and (13)) were obtained by numerical solution by a point-slope method on an IBM 653 electronic computer. The value of X at which the slope term $d(T/T_0)/dX$ attained a value of zero was taken as the generalized length of the plate

$$L = l\sqrt{\frac{\sigma E(10^9)}{kt}\left(\frac{T_0}{1000}\right)^3} \tag{15}$$

for the given value of T_l/T_0 and T_s^*/T_0. For the X-increment schedules used, a comparison between the exact solution of equation (14) and a corresponding numerical solution indicated that numerical values of T/T_0 were within -0.00002 of the exact values. The values of L determined for the finite plate were estimated to be correct to about 0.5 percent.

E-529

Heat-Transfer Relations

Heat radiated. - At any position X along the plate, the heat radiated from the surfaces from X = 0 to X = X can be obtained in either of two ways. First, equation (4) can be integrated directly in terms of X to give

$$Q_X = W\sqrt{\sigma Ekt(10^{15})\left(\frac{T_0}{1000}\right)^5}\int_0^X\left[\left(\frac{T}{T_0}\right)^4 - \left(\frac{T_s^*}{T_0}\right)^4\right]dX \tag{16}$$

or, for comparison purposes in terms of a heat-radiated quotient,

$$\frac{Q_X/W}{\sqrt{\sigma Ekt(10^{15})\left(\frac{T_0}{1000}\right)^5}} = \int_0^X\left(\frac{T}{T_0}\right)^4 dX - \left(\frac{T_s^*}{T_0}\right)^4 X \tag{17}$$

The total heat radiated from a plate of length l is then

$$\frac{Q_L/W}{\sqrt{\sigma Ekt(10^{15})\left(\frac{T_0}{1000}\right)^5}} = \int_0^L\left(\frac{T}{T_0}\right)^4 dX - \left(\frac{T_s^*}{T_0}\right)^4 L \tag{18}$$

Second, since only internal heat conduction is involved in the plate heat transfer, the heat radiated from the surfaces will be equal to the heat conducted into the plate from the source at x = 0 minus the heat conducted past the point x = x. This conducted heat is given by

$$Q_c = -ktW\left(\frac{dT}{dx}\right)_0 + ktW\left(\frac{dT}{dx}\right)_x = Q_X \tag{19}$$

or, in terms of T/T_0 and X,

$$Q_X = -ktWT_0\sqrt{\frac{\sigma E(10^9)}{kt}\left(\frac{T_0}{1000}\right)^3}\left\{\left[\frac{d(T/T_0)}{dX}\right]_0 - \left[\frac{d(T/T_0)}{dX}\right]_X\right\}$$

from which is obtained for the heat-radiated parameter

$$\frac{Q_X/W}{\sqrt{\sigma Ekt(10^{15})\left(\frac{T_0}{1000}\right)^5}} = -\left\{\left[\frac{d(T/T_0)}{dX}\right]_0 - \left[\frac{d(T/T_0)}{dX}\right]_X\right\} \tag{20}$$

The slopes of the temperature ratios at $X = 0$ and $X = X$ are obtained from equation (12) for $T/T_0 = 1.0$ and $T/T_0 = T/T_0$, so that

$$\frac{Q_X/W}{\sqrt{\sigma Ekt(10^{15})\left(\frac{T_0}{1000}\right)^5}} = 0.6325\left\{\sqrt{\left[1 - \left(\frac{T_l}{T_0}\right)^5\right] - 5\left(\frac{T_s^*}{T_0}\right)^4\left(1 - \frac{T_l}{T_0}\right)} - \sqrt{\left[\left(\frac{T}{T_0}\right)^5 - \left(\frac{T_l}{T_0}\right)^5\right] - 5\left(\frac{T_s^*}{T_0}\right)^4\left(\frac{T}{T_0} - \frac{T_l}{T_0}\right)}\right\} \tag{21}$$

The total heat radiated from the plate Q_L is obtained from equation (21) for $T/T_0 = T_l/T_0$, in which case the second radical becomes zero.

For the plate of infinite length, since $T_l = T_s^*$,

$$\frac{(Q_X/W)_\infty}{\sqrt{\sigma Ekt(10^{15})\left(\frac{T_0}{1000}\right)^5}} = 0.6325\left[\sqrt{1 + 4\left(\frac{T_s^*}{T_0}\right)^5 - 5\left(\frac{T_s^*}{T_0}\right)^4} - \sqrt{\left(\frac{T}{T_0}\right)^5 - 5\left(\frac{T_s^*}{T_0}\right)^4\left(\frac{T}{T_0}\right) + 4\left(\frac{T_s^*}{T_0}\right)^5}\right] \tag{22}$$

According to equation (22), the maximum heat radiated Q_∞ in this case will occur when the second radical is zero, that is, when $T/T_0 = T_s^*/T_0$, which, as expected, is the condition at infinity. Equation (22) also shows that, for the infinite plate, the effect of equivalent sink temperature on the maximum heat radiated can be indicated by the ratio

$$\frac{Q_\infty}{(Q_\infty)_{T_s^*=0}} = \sqrt{1 + 4\left(\frac{T_s^*}{T_0}\right)^5 - 5\left(\frac{T_s^*}{T_0}\right)^4} \tag{23}$$

where

$$\left(\frac{Q_\infty}{W}\right)_{T_s^*=0} = 0.6325\sqrt{\sigma Ekt(10^{15})\left(\frac{T_0}{1000}\right)^5} \tag{24}$$

Heat ratio. - In many instances, it is convenient to consider the local heat radiated in terms of its ratio to the terminal value for a given plate configuration. This ratio, called the heat ratio Q_X/Q_L, is given from equations (17) and (18) as

$$\frac{Q_X}{Q_L} = \frac{\displaystyle\int_0^X \left(\frac{T}{T_0}\right)^4 dX - \left(\frac{T_s^*}{T_0}\right)^4 X}{\displaystyle\int_0^L \left(\frac{T}{T_0}\right)^4 dX - \left(\frac{T_s^*}{T_0}\right)^4 L} \tag{25}$$

or, from equation (21), as

$$\frac{Q_X}{Q_L} = 1 - \sqrt{\frac{\left[\left(\frac{T}{T_0}\right)^5 - \left(\frac{T_l}{T_0}\right)^5\right] - 5\left(\frac{T_s^*}{T_0}\right)^4\left(\frac{T}{T_0} - \frac{T_l}{T_0}\right)}{\left[1 - \left(\frac{T_l}{T_0}\right)^5\right] - 5\left(\frac{T_s^*}{T_0}\right)^4\left(1 - \frac{T_l}{T_0}\right)}} \tag{26}$$

For the special case of the infinite plate and zero sink temperature, the heat ratio is obtained from equations (22) and (24) with $T_s^* = 0$ and with T/T_0 from equation (14). The result is

$$\frac{Q_X}{Q_\infty} = 1 - \left[\frac{1}{1 + 0.9487\ x\sqrt{\frac{\sigma E(10^9)}{kt}\left(\frac{T_0}{1000}\right)^3}}\right]^{5/3} \tag{27}$$

E-529

Radiating effectiveness. - A measure of the radiating effectiveness of the plate can be obtained by comparing the actual heat radiated from the plate to the heat that would be radiated if the plate were at a constant temperature equal to the source temperature. This isothermal heat radiated up to any distance x, designated by the symbol $\overline{Q}_x$, is obtained from integration of equation (4) for $T = T_0$ to give

$$\frac{\overline{Q}_x}{W} = \sigma E x (10^{12}) \left(\frac{T_0}{1000}\right)^4 \left[1 - \left(\frac{T_s^*}{T_0}\right)^4\right]$$

or, in terms of the generalized distance parameter X, from equation (17),

$$\frac{\overline{Q}_X/W}{\sqrt{\sigma E k t (10^{15}) \left(\frac{T_0}{1000}\right)^5}} = X\left[1 - \left(\frac{T_s^*}{T_0}\right)^4\right] \qquad (28)$$

Thus, the radiating effectiveness, given by the ratio $Q_X/\overline{Q}_X$ and designated by the symbol η, becomes, in general,

$$\eta = \frac{Q_X}{\overline{Q}_X} = \frac{\frac{1}{X}\int_0^X \left(\frac{T}{T_0}\right)^4 dX - \left(\frac{T_s^*}{T_0}\right)^4}{1 - \left(\frac{T_s^*}{T_0}\right)^4} \qquad (29)$$

or, by dividing equation (21) by equation (28),

$$\eta = \frac{0.6325}{X\left[1 - \left(\frac{T_s^*}{T_0}\right)^4\right]} \left\{\sqrt{\left[1 - \left(\frac{T_l}{T_0}\right)^5\right] - 5\left(\frac{T_s^*}{T_0}\right)^4\left(1 - \frac{T_l}{T_0}\right)} - \sqrt{\left[\left(\frac{T}{T_0}\right)^5 - \left(\frac{T_l}{T_0}\right)^5\right] - 5\left(\frac{T_s^*}{T_0}\right)^4\left(\frac{T}{T_0} - \frac{T_l}{T_0}\right)}\right\} \qquad (30)$$

For the infinite plate with nonzero sink temperature, since $T_l/T_0 = T_s^*/T_0$, the radiating effectiveness of equation (30) reduces to

$$\eta_\infty = \frac{0.6325}{X\left[1 - \left(\frac{T_s^*}{T_0}\right)^4\right]} \sqrt{1 + 4\left(\frac{T_s^*}{T_0}\right)^5 - 5\left(\frac{T_s^*}{T_0}\right)^4} - \sqrt{\left(\frac{T}{T_0}\right)^5 + 4\left(\frac{T_s^*}{T_0}\right)^5 - 5\left(\frac{T_s^*}{T_0}\right)^4\left(\frac{T}{T_0}\right)} \tag{31}$$

and, for the special case of zero sink temperature, to

$$\eta_\infty = \frac{0.6325}{X}\left[1 - \left(\frac{T}{T_0}\right)^{5/2}\right]$$

or, from equation (14), to

$$\eta_\infty = \frac{0.6325}{x\sqrt{\frac{\sigma E(10^9)}{kt}\left(\frac{T_0}{1000}\right)^3}}\left\{1 - \left[\frac{1}{1 + 0.9487\ x\sqrt{\frac{\sigma E(10^9)}{kt}\left(\frac{T_0}{1000}\right)^3}}\right]^{5/3}\right\} \tag{32}$$

The same result, of course, can be obtained from direct integration of equation (29) in conjunction with equation (14).

HEAT-TRANSFER RESULTS

The results and discussion for the heat-transfer characteristics of the fin plate will be presented in terms of the infinite-length plate and the finite-length plate. The results for the infinite plate are useful in establishing reference variations and in indicating the influence of the principal variables involved.

Infinite-Length Plate

Zero sink temperature. - Exact solutions for the heat-transfer characteristics of a fin plate of infinite length were obtained for the condition of zero sink temperature from equations (14), (27), and (32). Although single curves of the various heat-transfer characteristics can be obtained with the use of the generalized distance parameter X, variations were computed for a range of values of source temperature to show the trends involved. The resulting plots are given in figures 3(a), (b), and (c), which show respectively the variations of temperature ratio, heat ratio, and radiating effectiveness with distance from the heat source for a wide range of source temperatures. In figure 3(b) it is seen that, as a result of the general ability of a unit surface area to radiate more heat at higher temperatures, the plate can radiate most of its heat in successively shorter distances as source temperature is increased. As a

E-529

E-529

consequence, the plate temperature will fall off more rapidly with distance at the higher source temperatures, as indicated in figure 3(a). The steeper temperature profiles at the higher source temperatures also result in a reduced radiating effectiveness, as shown in figure 3(c), since the integrated value of $(T/T_0)^4$ in equation (29) will decrease as T_0 increases.

It is also noted that according to equations (14), (27), and (32) the physical distance x at which a given temperature ratio or heat ratio is attained will also be a function of plate thickness, emissivity, and thermal conductivity. However, in view of the powers involved in each variable, variations in x due to changes in these quantities should generally be less pronounced than the variations due to corresponding changes in source temperature.

Nonzero sink temperature. - Examples of the change in temperature-ratio profile that occurs for a fixed value of source temperature when the equivalent sink temperature varies are shown for illustrative purposes in figure 4. The effect of increasing sink temperature reduces the plate distance at which most of the heat is radiated. For purposes of generalization, the variation of plate heat-transfer characteristics with equivalent sink temperature ratio T_s^*/T_0 as obtained from the numerical solutions is shown in figure 5 in terms of the generalized distance parameter $X = x\sqrt{\frac{\sigma E(10^9)}{kt}\left(\frac{T_0}{1000}\right)^3}$. The temperature profiles are expressed as the ratio $(T - T_s^*)/(T_0 - T_s^*)$ in figure 5(a) in order to normalize the curves between the limits of zero and unity, and the heat ratio and radiating effectiveness as before are given in figures 5(b) and (c), respectively. (The curves for $T_s^*/T_0 = 0$ in fig. 5 represent all the curves in fig. 3.)

The effect of equivalent sink temperature ratio in reducing the effective radiating distance along the plate for the entire range of T_s^*/T_0 is clearly indicated in figures 5(a) and (b). The radiating effectiveness, however, as shown in figure 5(c), varies only little with sink temperature ratio. Actually, for values of X less than about 0.5, the effectiveness increases with increasing T_s^*/T_0, while, for values of X greater than about 1.25, a decrease is observed.

In order to obtain an understanding of the variation of radiating effectiveness with sink temperature ratio, it is noted from equation (29) that η will depend on the ratio of decrease of the numerator compared with the rate of decrease of the denominator as T_s^*/T_0 is increased. The numerator represents the difference between the average value of the fourth power of T/T_0 over the X increment in question and the base value $(T_s^*/T_0)^4$. This difference will therefore depend on the curvature of the temperature-ratio variation. If the temperature-ratio profiles were similar for all sink temperatures, numerator and denominator would increase proportionately, and the effectiveness would remain constant with sink temperature. However, for the infinite fin plate, as shown in figure 5(a), the temperature profiles are not similar. Thus, the variation in effectiveness with sink temperature ratio is a result of the nonsimilarity of the profiles. For values of X less than about 0.5 in figure 5(a), less change in curvature of the temperature variation occurs as sink temperature ratio is increased. The average value of $(T/T_0)^4$ will therefore tend to increase relative to the base value and produce an increase in effectiveness. On the other hand, for X greater than 1.25, the average value will tend to decrease relative to the base value as T_s^*/T_0 is increased, and thus a lower effectiveness will be obtained. In fact, the decreasing effectiveness with T_s^*/T_0 for large values of X can readily be established mathematically from equation (31). It has been shown that a maximum value of Q_X is attained theoretically at infinity. However, values of Q_X approaching the maximum are attained quite rapidly in many cases. Thus, the effectiveness tends to approach the variation given by $Q_\infty/\overline{Q}_\infty$. The maximum value of effectiveness is then obtained from equation (31) for $T/T_0 = T_s^*/T_0$ to be

E-529

$$\eta_\infty = \frac{0.6325}{X}\left[\frac{\sqrt{1 + 4\left(\frac{T_s^*}{T_0}\right)^5 - 5\left(\frac{T_s^*}{T_0}\right)^4}}{1 - \left(\frac{T_s^*}{T_0}\right)^4}\right] \tag{33}$$

The variation of effectiveness with sink temperature ratio at a fixed large value of X is readily established from substitution of several trial values.

As far as maximum heat radiated is concerned, figure 6 illustrates the variation with sink temperature as obtained from equation (23). It is seen that values of T_s^*/T_0 up to 0.4 can be attained with less than a 5-percent reduction in heat radiated. Also shown in the figure for comparison is the variation expected from a hypothetical plate of constant temperature throughout. The maximum heat ratio for the fin plate over the full range of T_s^*/T_0 is therefore not very much different than that for the isothermal plate.

Finite-Length Plate

Terminal values. - For the case of a fin plate of finite length, the temperature profiles will be altered somewhat compared to the infinite plate in order that the boundary condition of zero temperature slope at the end of the plate be met, as illustrated in figure 7. For large values of X, as indicated in figure 7, only small changes in the profile are necessary, since slope values for the infinite plate are small in this region. However, as X decreases and the slope for the infinite plate becomes larger, a greater difference between the terminal value of T/T_0 and the infinite plate value is required to produce the zero slope. However, since it is expected that the temperature ratios must approach unity at $X = 0$ in both cases, the difference in T/T_0 must then approach zero at the leading edge.

Variations of terminal temperature ratio $(T_l - T_s^*)/(T_0 - T_s^*)$ with generalized length parameter $l\sqrt{\frac{\sigma E(10^9)}{kt}\left(\frac{T_0}{1000}\right)^3}$ obtained from the numerical solutions are shown in figure 8 for a wide range of equivalent sink temperature ratios. The increase in temperature ratio at the end of a plate of finite length compared with that value for the infinite plate at the same distance can readily be obtained from a comparison of figures 8 and 5. As expected, maximum differences in temperature ratio are obtained in the region from about $L = 0.4$ to 0.8.

In view of the increased temperature along the finite plate compared to the infinite plate up to the same value of length, it is expected that the total heat radiated from the finite plate (Q_L) will be greater than that radiated from the infinite plate up to $x = l$ $(Q_{L,\infty})$. Also, the maximum difference in heat radiated should follow the maximum differences in temperature ratio. The comparison ratio $Q_L/Q_{L,\infty}$ obtained from the calculations is shown in figure 9. Also presented in the figure are lines of constant terminal temperature ratio as obtained from figure 8. The ratio of heat radiated is found to decrease with increasing sink temperature, since the difference between $(T/T_0)_l$ and $(T/T_0)_{l,\infty}$ decreased with increasing T_s^*/T_0. The effect of sink temperature is pronounced only for values of T_s^*/T_0 greater than about 0.2.

Since the ideal heat radiated (heat radiated at source temperature) is the same for both the finite and infinite plates, the increase in heat radiated for the finite plate should result in a greater radiating effectiveness for the finite plate, especially at the low values of X where $Q_L/Q_{L,\infty}$ is large. Calculated variations of radiating effectiveness with plate-length parameter are plotted in figure 10 to show this trend (as seen by comparison with fig. 5(c)). As for the infinite plate, the variation of radiating effectiveness with equivalent sink temperature is not large. Apparently, the effect of the nonsimilarity of temperature profiles for the finite plate is such as to give a decrease in effectiveness with increasing sink temperature at all values of X. Thus, the total heat radiated from a fin plate of given length can readily be determined from figure 10 and equation (28).

The effect of equivalent sink temperature on the magnitude of the heat radiated for a plate of given length is illustrated in figure 11. Here again, a less-than-5-percent reduction in sink temperature total heat radiated is obtained for values of T_s^*/T_0 up to about 0.4.

Variations along fin. - For the variations of temperature ratio and heat radiated along plates of finite length, it was found more practical to present the results in terms of derived analytical functions rather than in the required large number of individual plots. As can be seen in figure 7, the difference in temperature ratio between the finite-length plate and the infinite plate at the same value of X steadily increases as X increases to L. It was found that this difference can be closely approximated by a power variation such that the local-temperature-ratio variation for the finite length plate can be expressed as

$$\frac{T - T_s^*}{T_0 - T_s^*} = \left(\frac{T - T_s^*}{T_0 - T_s^*}\right)_\infty + \left[\Delta\left(\frac{T - T_s^*}{T_0 - T_s^*}\right)_l\right]\left(\frac{X}{L}\right)^n \tag{34}$$

where $\left[(T - T_s^*)/(T_0 - T_s^*)\right]_\infty$ is the temperature ratio for the infinite plate at the particular value of X in question (fig. 5(a)), and the term within the brackets represents the difference between the terminal value of temperature ratio (at the end of the plate) and the value of temperature ratio for the infinite plate at the same value of X (i.e., at $X = L$). Values of terminal temperature ratio are obtained from figure 8, and values of the exponent n deduced from the numerical solution as a function of terminal temperature ratio and equivalent sink temperature are given in figure 12. The maximum error in temperature ratio involved in the use of the power relation was about ±0.005.

The variation of heat radiated for a given value of T_l/T_0 and T_s^*/T_0 can be obtained from the local values of T/T_0 determined previously in conjunction with equation (21). In an alternate procedure, the heat variation can be calculated from the derived empirical relation

$$\frac{Q_X}{Q_{X,\infty}} = 1 + \left(\frac{Q_L}{Q_{L,\infty}} - 1\right)\left(\frac{X}{L}\right)^m \tag{35}$$

where $Q_{X,\infty}$ is the heat radiated up to the value X for the infinite plate (fig. 5(b) and eq. (22), or fig. 5(c) and eq. (28)), $Q_L/Q_{L,\infty}$ is the terminal heat ratio as given in figure 9, and deduced values of exponent m are given in figure 13. Maximum error involved in the use of equation (35) for the heat ratio $Q_X/Q_{X,\infty}$ was calculated to be about ± 0.005.

E-529

CL-3 back

PLATE VOLUME RELATIONS

Another factor important in the analysis of the fin plate as a radiator component is the relation between the plate volume and the plate heat-transfer characteristics. In particular, it is desirable to determine the values of the plate thickness, width, and length that will result in a minimum plate volume for a given amount of heat radiated. Such relations for minimum plate volume are necessary for optimizations of radiator configurations.

The volume of the plate is given by

$$V = tlW \tag{36}$$

and the heat radiated from the plate, as obtained from equations (19) and (10), is

$$Q_l = 632.5\sqrt{t}\,W\sqrt{Ek\sigma(10^9)\left(\frac{T_0}{1000}\right)^5}\sqrt{\left[1 - \left(\frac{T_l}{T_0}\right)^5\right] - 5\left(\frac{T_s^*}{T_0}\right)^4\left[1 - \left(\frac{T_l}{T_0}\right)\right]} \tag{37}$$

In equation (37), since there is a value of generalized length parameter L corresponding to each value of T_l/T_0 for a given T_s^*/T_0, the radical term will be a function of L. Thus, equation (37) can be expressed as

$$Q_l = 632.5\sqrt{t}\,W\sqrt{Ek\sigma(10^9)\left(\frac{T_0}{1000}\right)^5}\sqrt{f(L)} \tag{38}$$

where

$$f(L) = f(l/\sqrt{t}) = 1 - \left(\frac{T_l}{T_0}\right)^5 - 5\left(\frac{T_s^*}{T_0}\right)^4\left(1 - \frac{T_l}{T_0}\right) \tag{39}$$

The ratio of plate volume to heat radiated is obtained in terms of the physical variables l and t from equations (36) and (38) for fixed properties and source temperature as

$$\frac{V}{Q_l} = \frac{Cl\sqrt{t}}{f(l\sqrt{t})} \tag{40}$$

where C is a constant. It is seen immediately that a minimum of the ratio V/Q_l occurs when t goes to zero. The value of l required to make the ratio go to zero, however, is not obvious, since the quotient involving l goes to infinity as l goes to infinity and is indeterminate when l goes to zero. Furthermore, inasmuch as $f(L)$ is always finite, equation (38) shows that for a fixed value of heat radiated W must go to infinity as t goes to zero. A low-volume fin will therefore generally be characterized by small thickness and large width.

In practical cases of radiator design, however, it may be necessary to place some restriction on the allowable value of either t, l, or W. It may therefore be more useful for design purposes to determine the relations among the remaining variables that will produce the least volume for a given heat radiated when one variable is held fixed.

Fixed length. - If the plate length must be fixed and the thickness and width are free to vary, the relation for ratio of plate volume to heat radiated is obtained from equation (36) after substitution for W from equation (38) and for t from equation (15) as

$$\left(\frac{V}{Q_l}\right)_{l'} = \frac{l^2\left[f_{l'}(L)\right]}{632.5\ k\left(\frac{T_0}{1000}\right)} \tag{41}$$

where

$$f_{l'}(L) = \frac{1}{L\sqrt{f(L)}} \tag{42}$$

The volume function $f_{l'}(L)$ decreases continuously with L as shown in figure 14(a), indicating that, for least volume at fixed length l, the design value of L should be as large as possible. This result is to be expected, of course, since it was indicated previously that least volume results from least thickness, and an increase in L for this case can result only from a decrease in thickness (for constant k, E, and T_0).

The specific variation of the volume function curves in figure 14(a) is explained as follows: As L increases, the square root of the thickness, according to equation (15), will decrease linearly with L. However, for low magnitudes of L (say between 0.1 and about 1.0), the length function $f(L)$ as shown in figure 15 will increase with L but at a rate less than linear. Thus, some increase in W will be required by equation (38) to maintain Q_l constant. But, since t decreases with the square of L, a marked decrease in the product tW and therefore in the volume is to be expected. When L attains values that result in an essentially constant f(L), the required increase in W will become greater and the decrease in volume should tend to be less. Specifically, in the region where f(L) is essentially constant, tW should vary as 1/L. Thus, the volume function variation in figure 14(a) should approach a slope of unity as L gets very large.

E-529

The procedure for a design of fixed length is thus to determine t from the value of l and some practically selected large value of L consistent with other design requirements (meteoroid puncture, thermal stress, etc.). The plate width W is then determined from equation (38) for the desired heat transfer from the computed values of f(L) and t. It should be noted that fixed-length designs with large values of L will necessarily have low radiating effectivenesses.

Fixed thickness. - If the plate thickness is fixed and the length and width can vary, the appropriate expression for ratio of plate volume to heat radiated can be obtained from equations (36) after substitution for W from equation (38) and for l from equation (15) as

$$\left(\frac{V}{Q_l}\right)_t = \frac{t\left[f_t(L)\right]}{632.5 \quad \sigma E(10^9)\left(\frac{T_0}{1000}\right)^4} \tag{43}$$

where

$$f_t(L) = \frac{L}{\sqrt{f(L)}} \tag{44}$$

The volume function $f_t(L)$ plotted in figure 14(b) decreases with decreasing L and appears to approach its minimum value as L approaches zero. These results indicate that least fin volume at fixed thickness requires least length, since only $l = 0$ can give $L = 0$.

The reason for the specific variation of the curves in figure 14(b) is obtained from consideration of the length function f(L) shown in figure 15. As L decreases from high values of L, f(L) will remain essentially constant. At fixed t, as indicated by equation (38), W will also tend to remain constant for a given value of Q_l. Thus, the volume will vary only with l, which decreases linearly with L. As L decreases to the region where a marked decrease in f(L) occurs, a corresponding increase in W will be necessary to produce the required Q_l, and the fall-off in volume will be progressively diminished. Apparently, as L approaches zero, the required increase in W becomes sufficient to completely offset the effect of the decrease in l and result in a minimum of the function.

Thus, when fixed thickness is prescribed, a low value of l should be used, and the corresponding values of L and W for the desired heat transfer and thermal properties are determined respectively from equations (15) and (38). In practical radiator designs, however, there will be limitations to the minimum value of l that can be used because of spacing requirements for the tubes and because a sufficient ratio of l to t must be maintained to validate the one-dimensionality of the solutions.

Fixed width. - From the preceding two cases, it was seen that, theoretically, the fin plate for low volume will be characterized by very small l and very small t, and, consequently, the width W will be very large. In some instances, however, it may be necessary to restrict the value of W so that the ratio Q/W is fixed. In this case, the ratio of plate volume to heat radiated is obtained from equation (36) with substitution for l from equation (15) and for t from equation (38). The result is

$$\frac{V}{Q_l} = \frac{(Q_l/W)^2 f_W(L)}{(632.46)^3(\sigma^2 10^{18})kE^2\left(\frac{T_0}{1000}\right)^9} \qquad (45)$$

where

$$f_W(L) = \frac{L}{[f(L)]^{3/2}} \qquad (46)$$

Plots of the volume function $f_W(L)$ in figure 14(c) show that a definite minimum of the function exists for all values of sink temperature. Values of L for minimum plate volume range from about 0.92 at $T_s^*/T_0 = 0$ to about 0.76 at $T_s^*/T_0 = 0.9$.

The reason for the existence of a minimum in this case is clear when the variations with L of t, l, and f(L) are considered for the case of equation (45). As L decreases from large values of L, f(L) remains essentially constant, so that, for fixed Q_l/W in equation (38), t also shows little variation. The plate volume (eq. (36)) will therefore tend to decrease linearly with L since only the variation of l is effective in this region. As L decreases to the region where f(L) starts to decrease, an increase in t must occur in order to satisfy equation (38), and the reduction in volume tends to become progressively smaller. In the limit as L and f(L) go to zero, t must go to infinity, and (since nonzero values of l are allowed) the volume also will go to infinity. Thus, a minimum of the volume function must occur.

In a fixed-width design therefore, minimum volume will be obtained if t is selected as obtained from equation (38) for the desired value of Q_l/W and for $f(L)$ evaluated at the minimum value of L as found in figure 14(c). The plate length l is then determined from equation (15). Radiating effectiveness for these minimum values of L will vary from about 0.565 at $T_s^*/T_0 = 0$ to 0.61 at $T_s^*/T_0 = 0.9$ (see fig. 10).

For convenience, a plot of terminal temperature ratio $(T_l - T_s^*)/(T_0 - T_s^*)$ against length function L upon which lines of constant ratio of plate volume to minimum volume are drawn is illustrated in figure 16. Thus, for fixed-width designs, the closeness to minimum plate volume can readily be determined for any selected value of plate-length parameter or terminal temperature ratio.

APPLICATION OF RESULTS

The principal application of the preceding theoretical results for the radiant heat-transfer characteristics of a rectangular fin plate will be in the design of waste-heat radiators for satellite and space vehicles. Since the heat source in the problem has been prescribed uniform along the leading edge of the plate, the results will apply only to radiators in which internal heat is supplied by means of a condensing vapor. Several possible radiator configurations employing fin plates are illustrated in figure 1.

In the first illustration (fig. 1(a)), a single or double isolated fin plate is used to radiate heat supplied from the centerbody. In order that the derived results be valid in this case, it is necessary that the net heat transfer from the fin plate is not significantly affected by any adjacent vehicle structure. The second figure (fig. 1(b)) represents a single-surface continuous fin radiator composed of a series of adjacent fin plates with internally attached heat supply tubes at spacing $S' = 2l$. The dashed lines in the figure indicate that the fin plates and supply tubes may also be of integral rather than welded construction. The expected temperature variation along the fin is also shown in the figure. Such a radiator construction might be part of a vehicle outer skin.

The total surface area required to radiate a given amount of heat in this case can be given by, since $E = \epsilon$,

$$A = nS'W = \frac{Q}{\eta\epsilon\sigma\left(T_0^4 - T_s^{*4}\right)} \tag{47}$$

where n is the number of supply tubes and η is the radiating effectiveness as defined previously. The heat supplied can also be related to the tube flow area as follows:

$$Q = \omega H_c = \rho v H_c n \pi r_t^2 \tag{48}$$

where ρ, v, and ω are respectively the vapor density, flow velocity, and flow rate, H_c is the heat of condensation of the fluid, and r_t is the inner radius of the tube. Thus, for given tube and fin plate thicknesses, a radiator design of minimum total weight can be determined from the use of equations (47) and (48) and the derived results for radiating effectiveness and minimum plate volume. (In such calculations, it may be necessary to consider the inner surface of the fin and the exposed outer surfaces of the tubes to be thermally insulated.)

E-529

A double-surface continuous fin radiator is shown in figure 1(c). Similar relations and procedures can be established for this configuration to establish the spacing and overall width for minimum weight. Comparisons can then be made with other double-surface configurations such as the adjacent tube radiator of reference 1 and the fin and tube radiator described in reference 5. For the radiator of figure 1(c), it should be noted that, in an actual case, the heat radiated from the fin surfaces might be somewhat greater than the values determined from the current analysis since some heat will be radiated from the tube surfaces to the central region of the inner surface of the fin plate.

In another application to radiator design, the preceding results can be utilized to determine the elementary thermal stresses that will occur in the fin plates. According to reference 6, the thermal stresses in a rectangular fin plate of constant thickness can be calculated from integration of the plate temperature variations as determined herein. Thus, significant application of the results derived in this analysis can be found in radiator design studies.

Lewis Research Center
National Aeronautics and Space Administration
Cleveland, Ohio, September 2, 1959

APPENDIX A

FIN ENVIRONMENT

As indicated in the section entitled ANALYSIS, simple algebraic relations for the net heat transfer between the fin plate and an arbitrary environment can be obtained only for certain simplified situations. In general, environments that can be treated in this way will consist of several separated discrete surfaces each at uniform temperature and/or a large enclosing surface also of uniform temperature, and one or more concentrated heat sources. Examples of such environments are given in figure 17.

In order to obtain simple solutions for the net heat transfer to these environments, the following basic conditions must be met:

(1) The emissions and reflections from all surfaces are diffuse (i.e., Lambert's cosine law is obeyed).

(2) The view factors from a differential area on the fin surface to the discrete or enclosing surfaces are essentially constant over the fin surface.

Condition (1) is the necessary assumption normally adopted in radiation analyses for rough surfaces. Condition (2) is approached by surfaces that are either far removed from the fin (figs. 17(a) and (b)) or, if close to the fin, by surfaces that are extremely large compared to the fin (fig. 17(c)). For example, such a composite environment might be represented by a fin on a spaceship adjacent to another spaceship near a planet. Specific developments for each of the illustrative environments in figure 17 will now be considered.

Complete Enclosure

Relations are available for the net heat transfer between an enclosed body and an enclosure of uniform temperature T_s and area A_s if the variation over the enclosure of the view factor from a differential area on the enclosure surface to the body is comparatively small. This assumption, as well as condition (2), can be met exactly for such geometries as concentric spheres, coaxial cylinders, or a thin equatorial disk in a sphere. For these configurations, the net heat transfer between the enclosed body and the enclosure is given by

$$Q = \frac{\epsilon_b \sigma \left(T_b^4 - T_s^4\right) A_b}{1 + \epsilon_b \left(\frac{1}{\epsilon_s} - 1\right) \frac{A_b}{A_s}} \tag{A1}$$

These conditions, however, for practical purposes, can be closely approximated for a body and enclosure for which the distance between the two surfaces does not vary much over the surfaces. For a radiating fin plate, this condition will be approached if the fin is centrally located within a very large enclosure (fig. 17(a)). The equation for net heat transfer between the fin strips and the sink surface then becomes, since $A_s \gg A_f$,

$$dQ = (\epsilon_{f1} + \epsilon_{f2})\sigma\left(T_f^4 - T_s^4\right)W\ dx \tag{A2}$$

In reality, such an environment might be represented by a fin plate at or near the center of a large furnace or other large enclosing structure.

Although not a true enclosing surface, outer space can be regarded as a black body enclosure of effective temperature T_s obeying equation (A2). For practical purposes, however, the sink temperature of space (of the order of about 40° R) can normally be neglected in radiator studies.

Discrete Surfaces with Concentrated Heat Source

The case of a fin environment consisting of one or more discrete surfaces and a concentrated heat source is illustrated in figure 17(b). Heat-transfer relations for this environment can readily be formulated if the additional condition is imposed that the view factors from the discrete surface to the fin strip are sufficiently small that multiple reflections from these surfaces will be negligible. (This condition will be approached if the ratio of the distance between the fin and the discrete surface to the fin length is large.) In many cases of radiator design, it may be sufficient to neglect reflections entirely; however, for completeness, the analysis will include first reflections from all heat sources.

The net heat loss from the fin to the environment is obtained as the difference between the heat emitted by the fin and the heat absorbed by the fin from the environment bodies. The heat emitted from the surfaces f1 and f2 of the fin strip $W\ dx$ is given by

$$dQ_f = (\epsilon_{f1} + \epsilon_{f2})\sigma T_f^4 W\ dx \tag{A3}$$

Heat will be received by the fin strips from direct emissions from the discrete surfaces and the concentrated heat source and from reflections from the discrete surfaces of heat emitted from the fin, the concentrated heat source, and other discrete surfaces. However, according to the scope of the analysis, only first reflections need be considered. If the environment is enclosed by outer space, there will also be a heat return from space, but, as indicated previously, this heat is comparatively insignificant.

E-529

CL-4 back

Discrete-surface emission. - The heat received by a fin strip from direct emission from a discrete surface of area A_i, uniform temperature T_i^4, and emissivity ϵ_i is

$$dQ_{i-f} = \epsilon_i \sigma T_i^4 A_i F_{i-f} \alpha_{fi} \tag{A4}$$

The double subscript fi for the total absorptivity in equation (A4) is used to indicate that the magnitude of α is based on the characteristics of both the fin surface (symbol f) and the incoming radiation from the discrete surface (symbol i). This distinction is necessary since, by definition,

$$\alpha_{fi} \equiv \frac{\int_0^\infty \alpha_{\lambda f} e_{\lambda i} \, d\lambda}{\int_0^\infty e_{\lambda i} \, d\lambda} \tag{A5}$$

where $e_{\lambda i}$ is the incident energy contained in the spectral increment $d\lambda$. Thus, the magnitude of α will depend on the characteristics and temperature of the absorbing surface and on the temperature of the original emitting body. Then, since temperature and emissivity are constant over the respective fin strip and discrete surfaces, equation (A4) can be expressed in terms of the fin strip area and the view factor from fin strip to discrete surface according to the reciprocity theorem.

$$A_i F_{i-f} = A_f F_{f-i} \tag{A6}$$

as

$$dQ_{i-f} = \epsilon_i \sigma T_i^4 A_f F_{f-i} \alpha_{fi} = \epsilon_i \sigma T_i^4 F_{f-i} \alpha_{fi} W \, dx \tag{A7}$$

Discrete-surface reflection. - The heat originating from another surface j that is reflected from the discrete surface i to the fin and is absorbed by the fin is given by

$$dQ_{j-f} = \left(\epsilon_j \sigma T_j^4 A_j F_{j-i}\right) \rho_{ij} F'_{i-f} \alpha_{fj} \tag{A8}$$

where F'_{i-f} is the view factor from the portion of surface i that is illuminated by the radiation from surface j (for a flat surface, $F'_{i-f} = F_{i-f}$), and where

$$\rho_{ij} \equiv \frac{\int_0^\infty \rho_{\lambda i} e_{\lambda f}\, d\lambda}{\int_0^\infty e_{\lambda j}\, d\lambda} \tag{A9}$$

Since no change in spectral distribution is assumed to occur after a reflection, the fin absorptivity α_{fj} is based on the radiation spectrum of the original emitting surface j. Equation (A8), from the reciprocity theorem for $A_j F_{j-i}$ and then for $A_i F'_{i-f}$, can then be expressed as

$$dQ_{j-f} = \epsilon_j \sigma T_j^4 F_{i-j} \rho_{ij} F'_{f-i} \alpha_{fj} W\, dx \tag{A10}$$

Heat-source radiation. - The heat received by the fin strip from direct and reflected heat-source (solar) radiation is

$$dQ_{S-f} = S(\cos\,\theta_{fS})\alpha_{fS} W\, dx + SA'_i \rho_{iS} F'_{i-f} \alpha_{fS} \tag{A11}$$

where A'_i is the projected area of a discrete surface normal to the incident radiation from the heat source, and F'_{i-f} is the view factor from the portion of the surface illuminated by the heat source to the fin strip. As before, equation (A11) can be expressed in terms of fin area as

$$dQ_{S-f} = S(\cos\,\theta_{fS})\alpha_{fS} W\, dx + S\left(\frac{A'_i}{A_i}\right)\rho_{iS} F'_{f-i} \alpha_{fS} W\, dx \tag{A12}$$

where, for the special case of a flat discrete surface,

$$\frac{A'_i}{A_i} = \cos\,\theta_{iS} \tag{A13}$$

Fin reflections. - Inasmuch as the temperature varies over the surface of the fin, the exact expression for the heat originating from the fin surface that is reflected back to the fin strip surface will involve integration of T_f^4 over the length of the fin. However, for the purposes of this analysis, it should be sufficient to consider an effective average value for the fourth-power variation of the fin temperature, denoted by $\overline{T_f^4}$, such that an equivalent algebraic expression can be used. In this case, the heat received by the fin after reflection from a discrete surface will be given by

$$dQ_{f-f} = \epsilon_f \sigma \overline{T}_f^4 (lW) F_{\overline{f}-i} \rho_{if} F_{i-f} \alpha_{ff} \tag{A14}$$

From the reciprocity theorem for $(lW)F_{\overline{f}-i}$, equation (A14) becomes

$$dQ_{f-f} = \epsilon_f \sigma \overline{T}_f^4 A_i F_{i-\overline{f}} F_{i-f} \rho_{if} \alpha_{ff}$$

or, again, from reciprocity for $A_i F_{i-f}$,

$$dQ_{f-f} = \epsilon_f \sigma \overline{T}_f^4 F_{i-\overline{f}} F_{f-i} \rho_{if} \alpha_{ff} W \, dx \tag{A15}$$

E-529

In an actual calculation, it will be necessary to assume a trial value for $\overline{T}_f^4$ in equation (A15) based on T_0 and T_l. A later check on the assumed value can be made, but it is doubtful whether any recalculation based on a more exact value for $\overline{T}_f^4$ will generally be necessary.

Net radiation. - The net heat lost by both fin surfaces is then obtained as the difference between equation (A3) and the summation of equations (A7), (A10), (A12), and (A15) for all discrete surfaces, or

$$\begin{aligned} dQ = (\epsilon_{f1} + \epsilon_{f2}) \sigma T_f^4 W \, dx - \Bigg\{ & \sum_{i=1}^{N_d} \epsilon_i T_i^4 F_{f1-i} \alpha_{f1i} + \sum_{i=1}^{N_d} \epsilon_i T_i^4 F_{f2-i} \alpha_{f2i} \\ & + \sum_{i=1}^{N_d} \sum_{j=1}^{N_d} \left(\epsilon_j T_j^4 F_{i-j} \rho_{ij} \alpha_{f1j}\right) F'_{f1-i} + \sum_{i=1}^{N_d} \sum_{j=1}^{N_d} \left(\epsilon_j T_j^4 F_{i-j} \rho_{ij} \alpha_{f2j}\right) F'_{f2-i} \\ & + \frac{S}{\sigma} \left[(\cos \theta_{f1S}) \alpha_{f1S} + (\cos \theta_{f2S}) \alpha_{f2S} + \sum_{i=1}^{N_d} \left(\frac{A'}{A}\right)_i \rho_{iS} F'_{f1-i} \alpha_{f1S} \right. \\ & \left. + \sum_{i=1}^{N_d} \left(\frac{A'}{A}\right)_i \rho_{iS} F'_{f2-i} \alpha_{f2S} \right] + \epsilon_{f1} \overline{T}_{f1}^4 F_{i-\overline{f1}} F_{f1-i} \rho_{if1} \alpha_{f1f1} \\ & + \epsilon_{f2} \overline{T}_{f2}^4 F_{i-\overline{f2}} F_{f2-i} \rho_{if2} \alpha_{f2f2} \Bigg\} \sigma W \, dx \end{aligned} \tag{A16}$$

In a practical calculation, the need for including many of the brace terms in equation (A16) will depend largely on the expected magnitude of the reflectivities and the view factors involved. The view factor from the fin strip to a discrete surface is defined specifically as

$$F_{f-i} = \frac{1}{A_f} \int_{A_f} \int_{A_i} \frac{\cos \beta_f \cos \beta_i \, dA_i \, dA_f}{\pi r^2} \tag{A17}$$

where r is the distance vector from the fin strip to a point on the discrete surface, β_f is the angle between r and the normal to the fin surface, and β_i is the angle between r and the normal to the discrete surface. A preliminary rough estimate of F_{f-i} can be obtained, however, if average values of β_f, β_i, and r are considered to give

$$F_{f-i} = \frac{\overline{\cos} \beta_f \overline{\cos} \beta_i A_i}{\pi \bar{r}^2} \tag{A18}$$

Thus, the maximum value of view factor will be of the order of

$$F_{f-i} = \frac{1}{\pi} \frac{A_i}{\bar{r}^2} \tag{A19}$$

<u>Example</u>. - A special case of interest covered by these relations is given by a fin plate orbiting around the earth such that one surface of the fin is parallel to the surface of the earth, and the other surface is exposed to the sun (fig. 17(c)). For this environment, equation (A16) reduces to, with subscript e used for the earth,

$$\begin{aligned} dQ = (\epsilon_{f1} + \epsilon_{f2})\sigma T_f^4 W \, dx - \Bigg\{ & \epsilon_e T_e^4 F_{f2-e} \alpha_{f2e} \\ & + \frac{S}{\sigma} \left[(\cos \theta_{f1S}) \alpha_{f1S} + \left(\frac{A'}{A}\right)_e \rho_{eS} F'_{f2-e} \alpha_{f2S} \right] \\ & + \epsilon_{f2} \overline{T}_{f2}^4 F_{e-\overline{f}2} F_{f2-e} \rho_{ef2} \alpha_{f2f2} \Bigg\} \sigma W \, dx \end{aligned} \tag{A20}$$

Since the earth's surface will be effectively flat and of infinite area with respect to the fin, both F_{f2-e} and F'_{f2-e} can be taken equal to unity, $F_{e-\overline{f}2}$ can be taken equal to zero, and $(A'/A)_e$ can be taken as

E-529

$\cos\,\theta_{f1S}$. Then, for equal thermal properties on both fin surfaces, equation (A20) becomes

$$dQ = \left\{2\epsilon_f\sigma T_f^4 - \left[\epsilon_e\sigma T_e^4\alpha_{fe} + S(\cos\,\theta_{fS})\alpha_{fS}(1 + \rho_{eS})\right]\right\}W\,dx \qquad \text{(A21)}$$

If, as a further simplification, the earth can be regarded as a black body so that $\epsilon_e = 1$ and $\rho_{eS} = 0$, the equation is reduced to

$$dQ = \left\{2\epsilon_f\sigma T_f^4 - \left[\sigma T_e^4\alpha_{fe} + S(\cos\,\theta_{fS})\alpha_{fS}\right]\right\}W\,dx \qquad \text{(A22)}$$

The fractional error involved in neglecting the earth sink temperature and solar radiation for this configuration is then

$$\text{Error} = \frac{1}{2}\left[\left(\frac{\alpha_{fe}}{\epsilon_f}\right)\left(\frac{T_e}{T_f}\right)^4 + \left(\frac{\alpha_{fS}}{\epsilon_f}\right)\frac{S(\cos\,\theta_{fS})}{\sigma T_f^4}\right] \qquad \text{(A23)}$$

Calculated variations of this error with fin temperature are shown in figure 18 for $(\alpha_{fe}/\epsilon_f) \approx (\alpha_{fS}/\epsilon_f) \approx 1$, which might represent a worst case for radiators. It is seen that the effects of solar radiation and earth sink temperature will be less than 5 percent for fin temperatures greater than about 1000° F.

General Form

For the environment cases considered herein, it is noted that, if the terms modifying σT_f^4 in equations (A2), (A16), and (A20) were designated by the symbol E, and if the terms within the braces of each equation were designated by the symbol B, then the net radiant heat loss from the plate can be represented in the form

$$dQ = \sigma\left\{ET_f^4 - B\right\}W\,dx \qquad \text{(A24)}$$

or

$$dQ = \sigma E\left\{T_f^4 - \frac{B}{E}\right\}W\,dx \qquad \text{(A25)}$$

Furthermore, if

$$\frac{B}{E} \equiv T_s^{*4} \qquad \text{(A26)}$$

then the heat loss relation becomes simply

$$dQ = \sigma E\left(T_f^4 - T_s^{*4}\right)W\ dx \tag{A27}$$

where T_s^* is called the equivalent sink temperature for the particular environment as defined by equation (A26). The quantity E is evaluated from the term modifying σT_f^4 in the heat-loss equation, and the quantity B is obtained from an evaluation of the brace term (i.e., all the terms that do not modify σT_f^4). For the three cases cited herein, E is equal to the sum of the emissivities on the two fin surfaces, that is, E-529

$$E = \epsilon_1 + \epsilon_2 \tag{A28}$$

However, this may not be so in all possible cases.

In a similar manner, a heat-loss relation in the form of equation (A27) can be established for other possible environments that meet the qualifications and assumptions listed. It should be noted, however, that equation (A27) will not be valid for environmental surfaces whose distance from the fin is less than about an order of magnitude greater than the length of the fin.

In equations (A2) and (A16), since the fin thermal properties are assumed constant with temperature, all ϵ_f and α_f values will be constant with x. Furthermore, since the fin plate is flat, θ_f will also be effectively constant with x. Then, with the view factors between the environment surfaces and the fin prescribed effectively constant over the fin (condition (2)), all quantities except T_f in these equations will be independent of x. Thus, E and T_s^* can be treated as constants in any integration of equation (A27).

E-529

CL-5

APPENDIX B

RADIATION LOSS FROM PLATE EDGE

If the plate end edge is exposed to an environment, a net radiant heat loss will occur from the end surface area Wt (fig. 2) according to the relation analogous to equation (5):

$$Q_l = \sigma E_l\left(T_l^4 - T_{s,l}^{*4}\right)Wt \tag{B1}$$

where the equivalent sink temperature with respect to the end edge $T_{s,l}^*$ need not necessarily be equal to T_s^* for the fin surfaces. This heat must be supplied internally from the plate by conduction such that

$$Q_{c,l} = -ktW\left(\frac{dT}{dx}\right)_l = Q_l \tag{B2}$$

It is thus obtained that, for this case,

$$\left(\frac{dT}{dx}\right)_l = -\frac{\sigma E_l}{k}\left(T_l^4 - T_{s,l}^{*4}\right) \tag{B3}$$

The constant of integration, from equations (7) and (B3) is then, after factoring terms,

$$C_l = -\frac{2\sigma E}{5kt}\left(T_l^4 - 5T_s^{*4}T_l\right)\left[1 - \frac{5\sigma E_l^2 t}{2kE}\,\frac{\left(T_l^4 - T_{s,l}^{*4}\right)^2}{\left(T_l^5 - 5T_s^{*4}T_l\right)}\right] \tag{B4}$$

The terms before the brackets, however, are equal to the constant of integration for zero edge heat loss as given by equation (8). Thus, equation (B4) can be expressed as

$$C_l = C\left\{1 - \frac{5\sigma E_l^2 t T_0^3}{2kE}\left(\frac{T_l}{T_0}\right)^3 \frac{\left[1 - \left(\frac{T_{s,l}^*}{T_l}\right)^4\right]^2}{\left[1 - 5\left(\frac{T_s^*}{T_l}\right)^4\right]}\right\} \tag{B5}$$

The approximate error in the constant of integration defined as $(C_l - C)/C$ is then

$$\text{Error} = -\frac{5\sigma E_l^2 t T_0^3}{2kE}\left(\frac{T_l}{T_0}\right)^3 \frac{\left[1 - \left(\frac{T^*_{s,l}}{T_l}\right)^4\right]^2}{\left[1 - 5\left(\frac{T^*_s}{T_l}\right)^4\right]} \tag{B6}$$

Now, since $E_l = \epsilon_l$ and $E = \epsilon_1 + \epsilon_2$, if ϵ is constant throughout,

$$\frac{E_l^2}{E} = \frac{\epsilon^2}{2\epsilon} = \frac{2\epsilon}{4} = \frac{E}{4} \tag{B7}$$

Substitution in equation (B6) then yields, after further manipulation:

$$\text{Error} = -\frac{5}{8}\left[l^2\left(\frac{\sigma E}{kt}\right)\left(\frac{T_0}{1000}\right)^3 10^9\right]\left(\frac{t}{l}\right)^2\left(\frac{T_l}{T_0}\right)^3 \frac{\left[1 - \left(\frac{T^*_{s,l}}{T_l}\right)^4\right]^2}{\left[1 - 5\left(\frac{T^*_s}{T_l}\right)^4\right]} \tag{B8}$$

The term within the first bracket, however, according to equation (15), is equal to the square of the generalized length parameter L. Thus, the error becomes

$$\text{Error} = -\frac{5}{8}\,\frac{L^2(T_l/T_0)^3}{(l/t)^2}\,\frac{\left[1 - \left(\frac{T^*_{s,l}}{T_l}\right)^4\right]^2}{\left[1 - \left(\frac{T^*_s}{T_l}\right)^4\right]} \tag{B9}$$

It is also noted that a maximum value of the ratio of the bracket terms is obtained when $T^*_s = T^*_{s,l} = 0$. Furthermore, since T_l/T_0 is a function of L, the product $L^2(T_l/T_0)^3$ can be evaluated as a function of L as shown in figure 19. It appears, from figure 19, that the function approaches a maximum value of about say 0.6. Thus, the maximum error is simply

$$\text{Max. error} \approx -\frac{0.375}{(l/t)^2} \tag{B10}$$

A plot of maximum error as given by equation (B10) is shown in figure 20. It is seen that the maximum error in the constant of integration resulting from the neglecting of the end edge heat loss will be less than 5 percent for values of length-to-thickness ratio as low as 3. Therefore, since it is most unlikely that fin plates will be designed for ratios of l/t of about 3 or less, it can be concluded that the effects of any heat loss from an exposed end edge can be neglected.

E-529

CL-5 back

REFERENCES

1. English, Robert E., Slone, Henry O., Bernatowicz, Daniel T., Davison, Elmer H., and Lieblein, Seymour: A 20,000-Kilowatt Nuclear Turboelectric Power Supply for Manned Space Vehicles. NASA MEMO 2-20-59E, 1959.

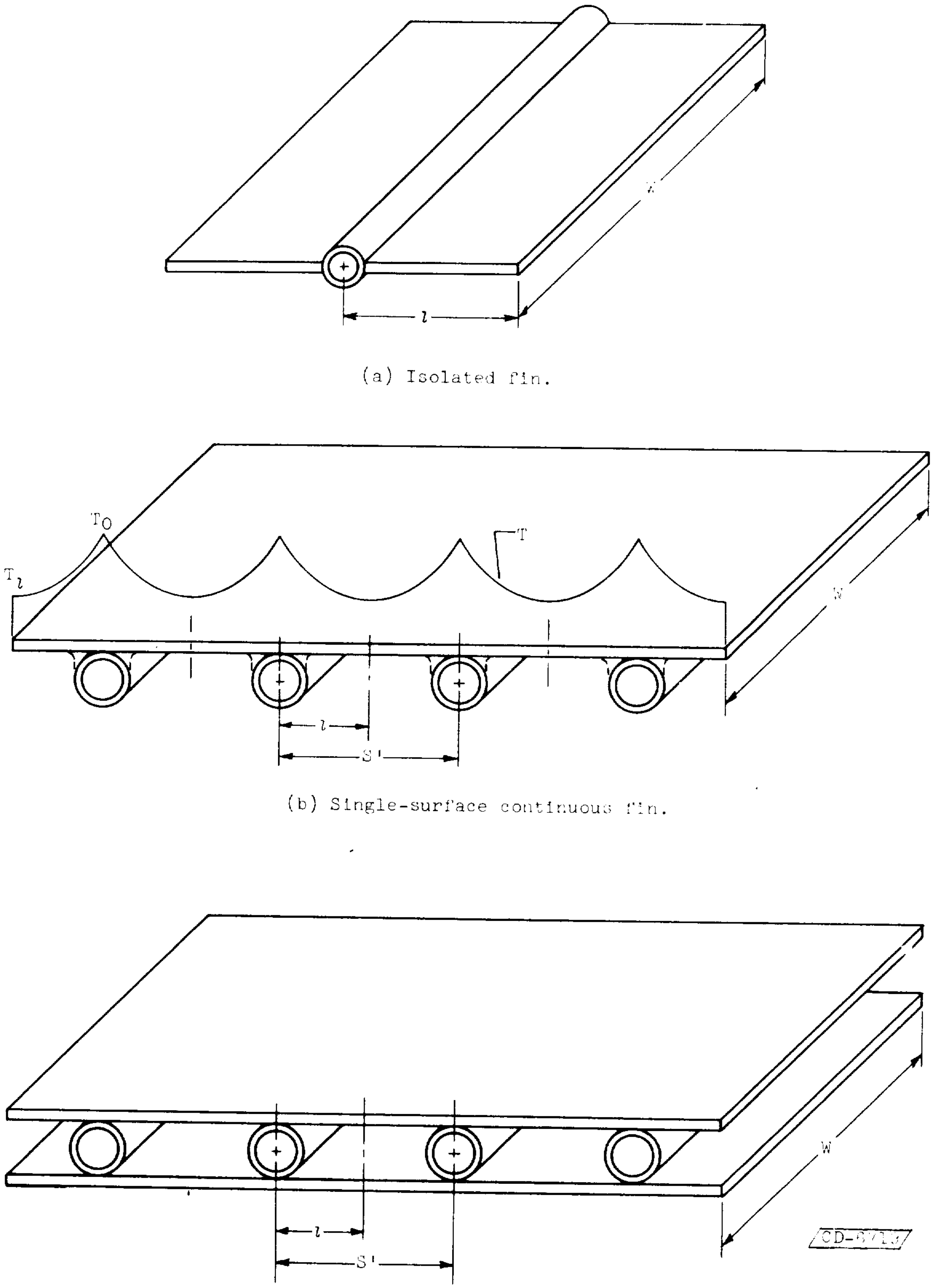
2. Mackay, D. B., and Leventhal, E. L.: Radiant Heat Transfer from a Flat Plate Uniformly Heated on One Edge. North American Aviation, Inc., MD 58-187, Aug. 13, 1958.

3. Carslaw, H. S., and Jaeger, J. C.: Conduction of Heat in Solids. Univ. Press (Oxford), 1950.

4. Jakob, Max: Heat Transfer. Vol. II. John Wiley & Sons, Inc., 1957.

5. Callinan, Joseph P., and Berggren, Willard P.: Some Radiator Design Criteria for Space Vehicles. Paper 59-AV-29, ASME, Mar. 1959.

6. Timoshenko, S.: Theory of Elasticity. McGraw-Hill Book Co., Inc., 1934, p. 209.



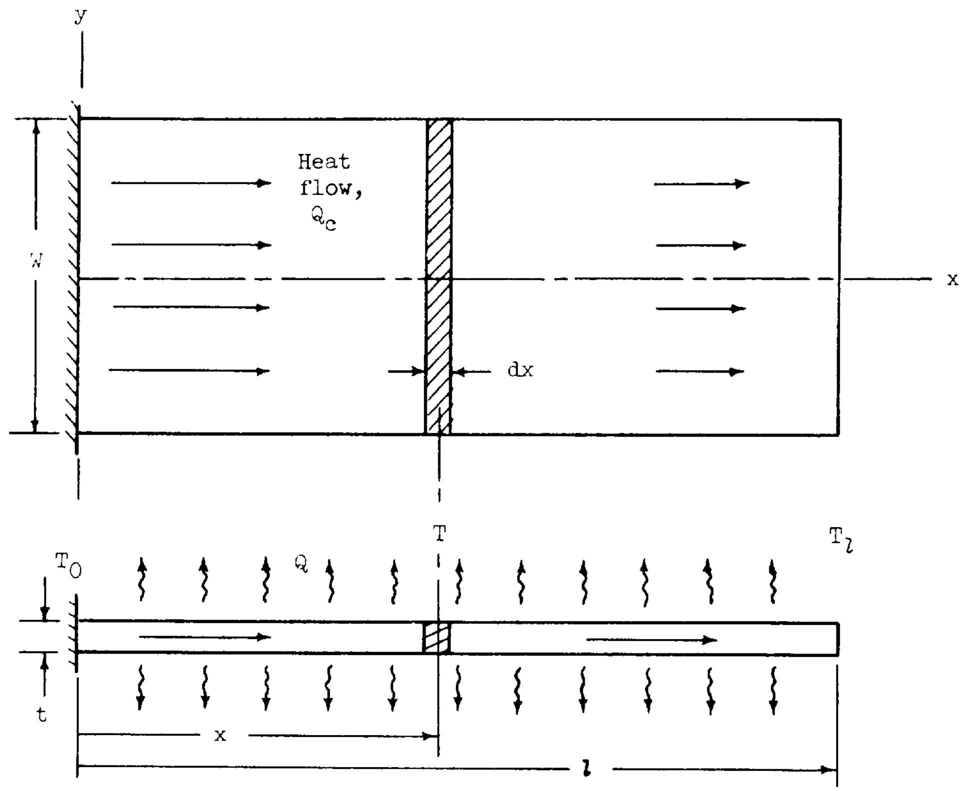
(a) Isolated fin.

(b) Single-surface continuous fin.

(c) Double-surface continuous fin.

Figure 1. - Radiator configurations involving rectangular fin plates.

E-529



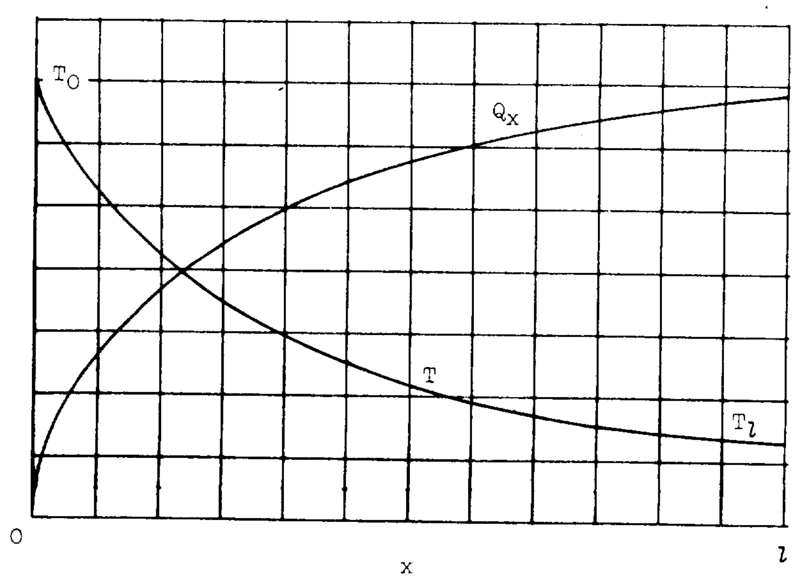
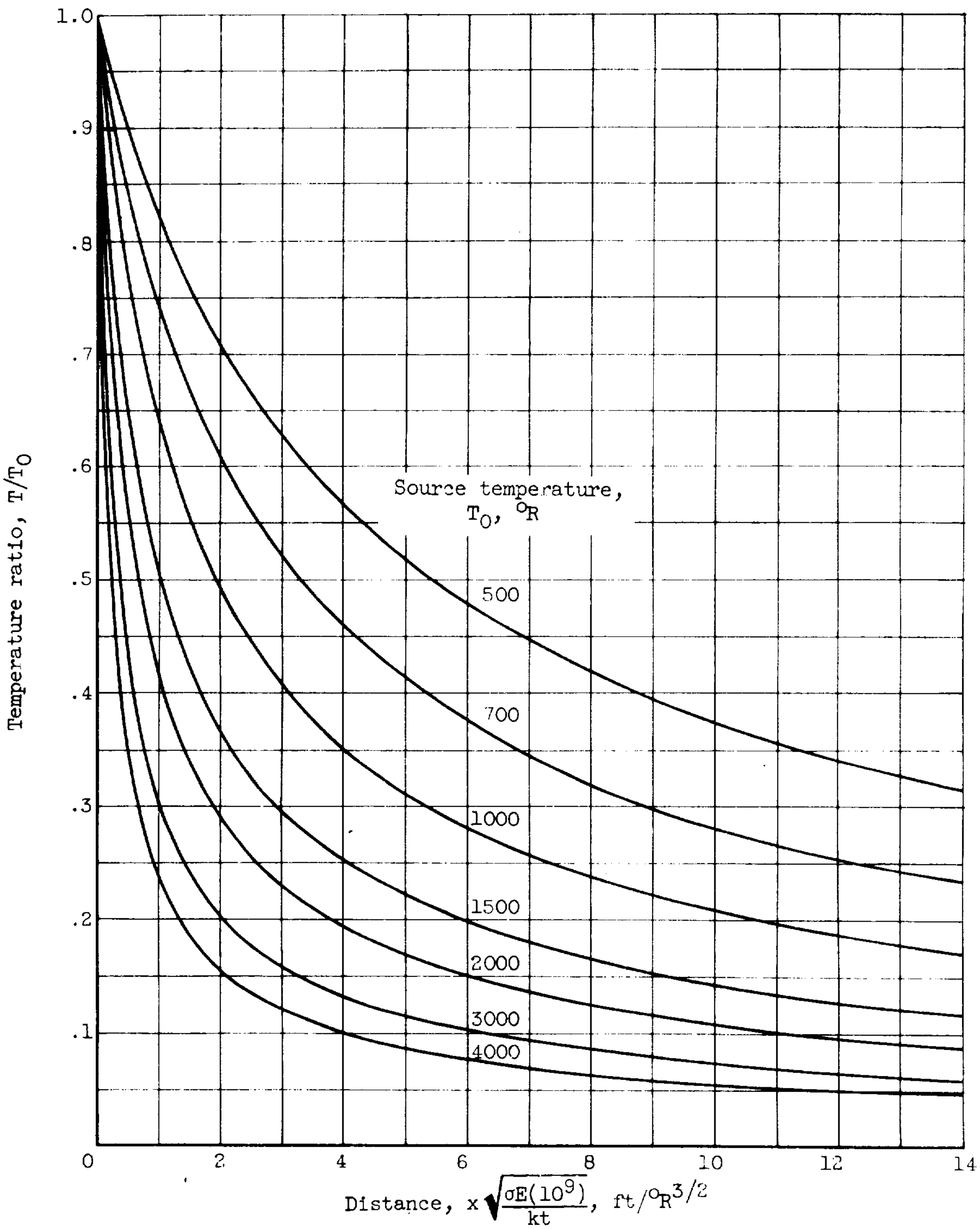


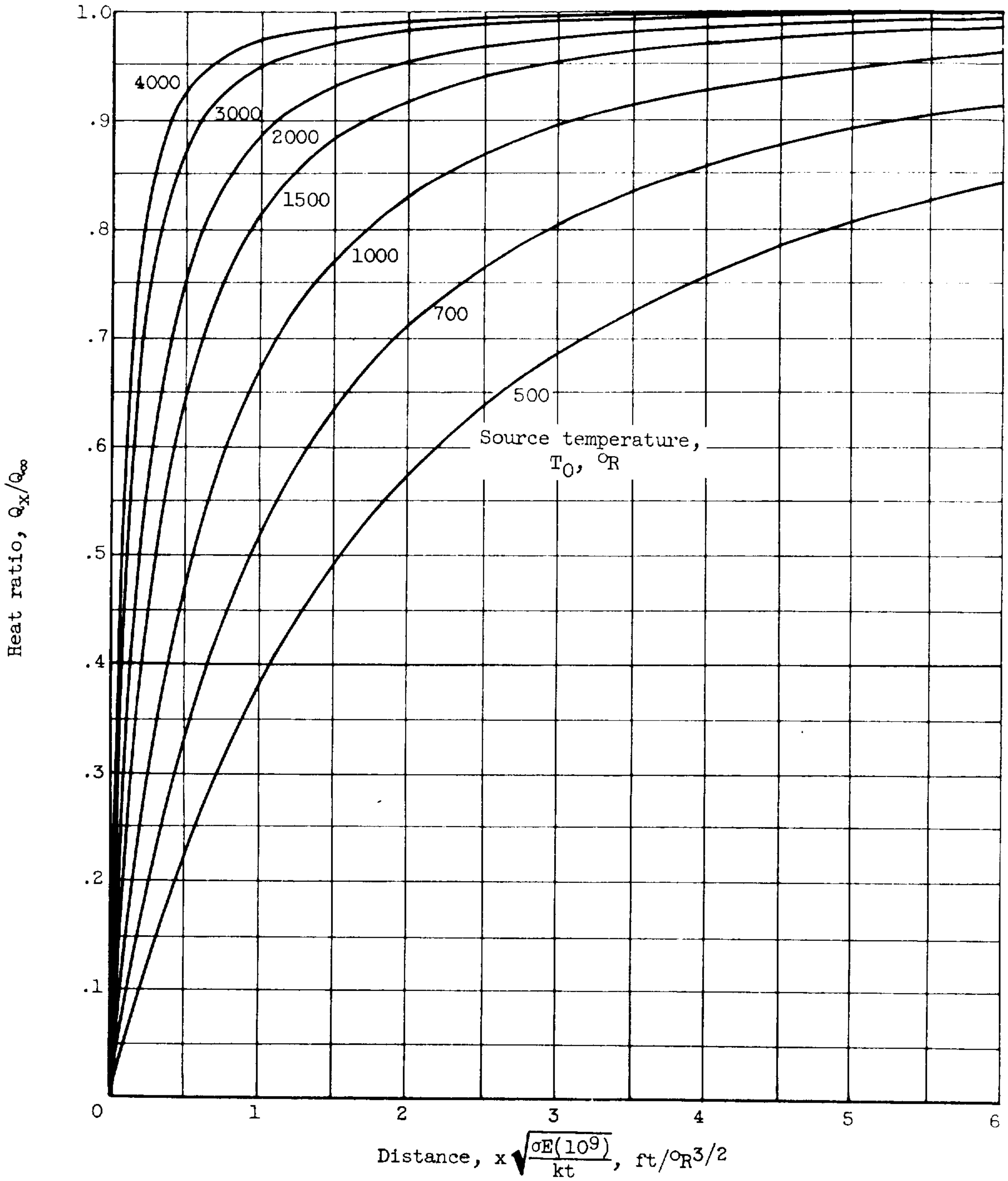
Figure 2. - Schematic diagram of flat fin plate showing expected variation of surface temperature and integrated radiant heat transfer.



(a) Variation of temperature ratio with distance from heat source.

Figure 3. - Heat-transfer characteristics of infinitely long plate at zero sink temperature.

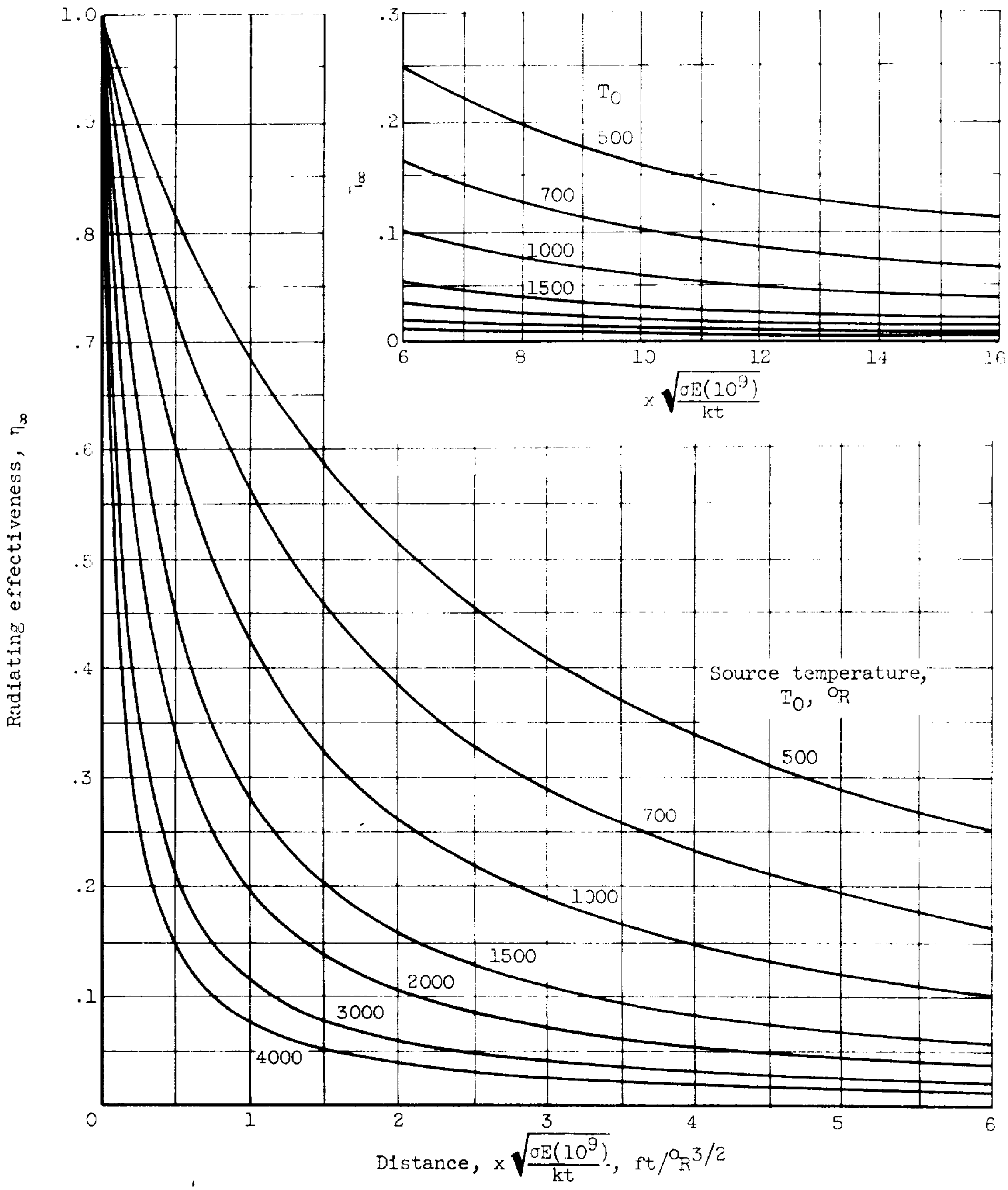
E-529



(b) Variation of heat ratio with distance from heat source.

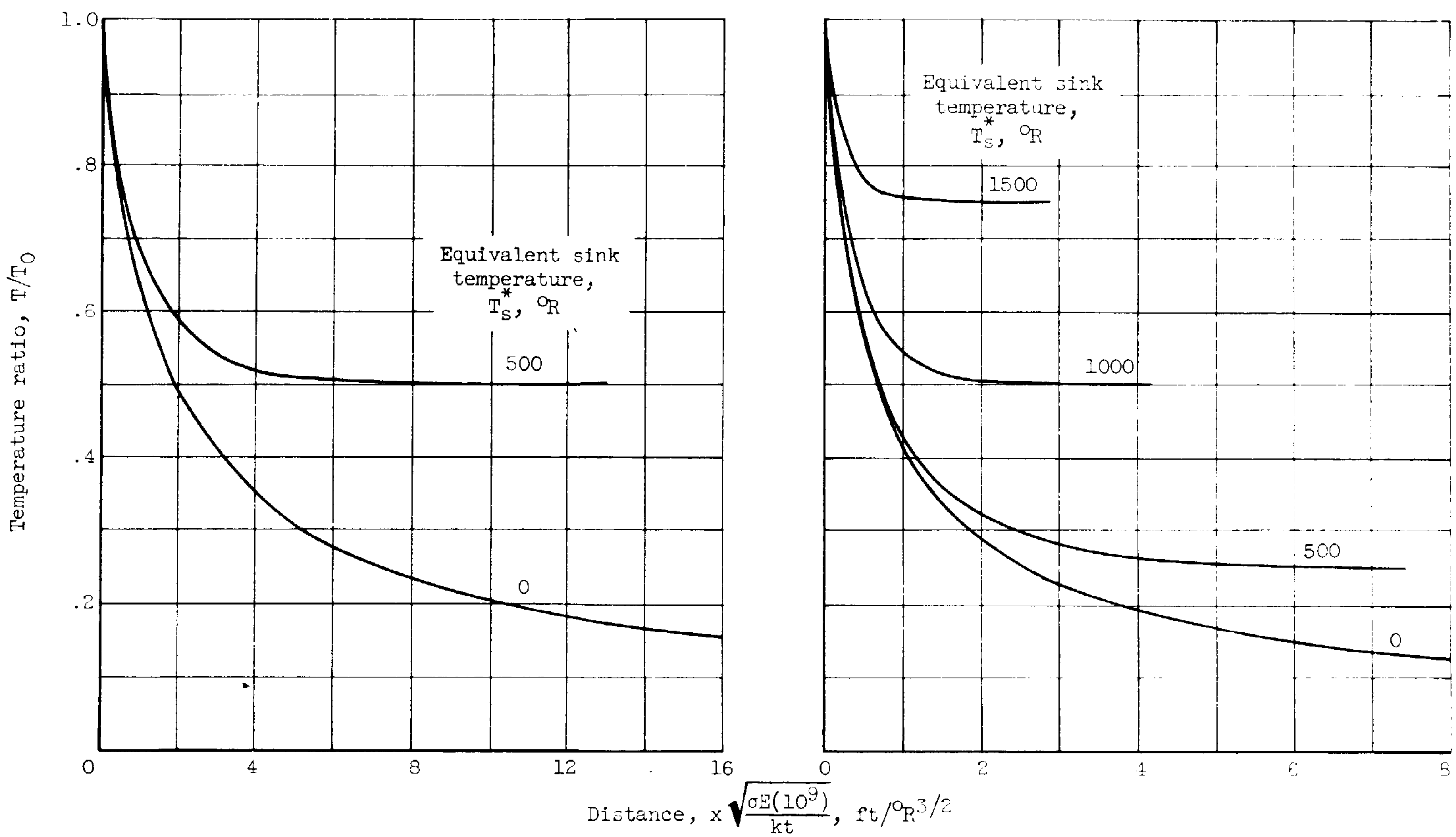
Figure 3. - Continued. Heat-transfer characteristics of infinitely long plate at zero sink temperature.

E-529



(c) Variation of radiating effectiveness with distance from heat source.

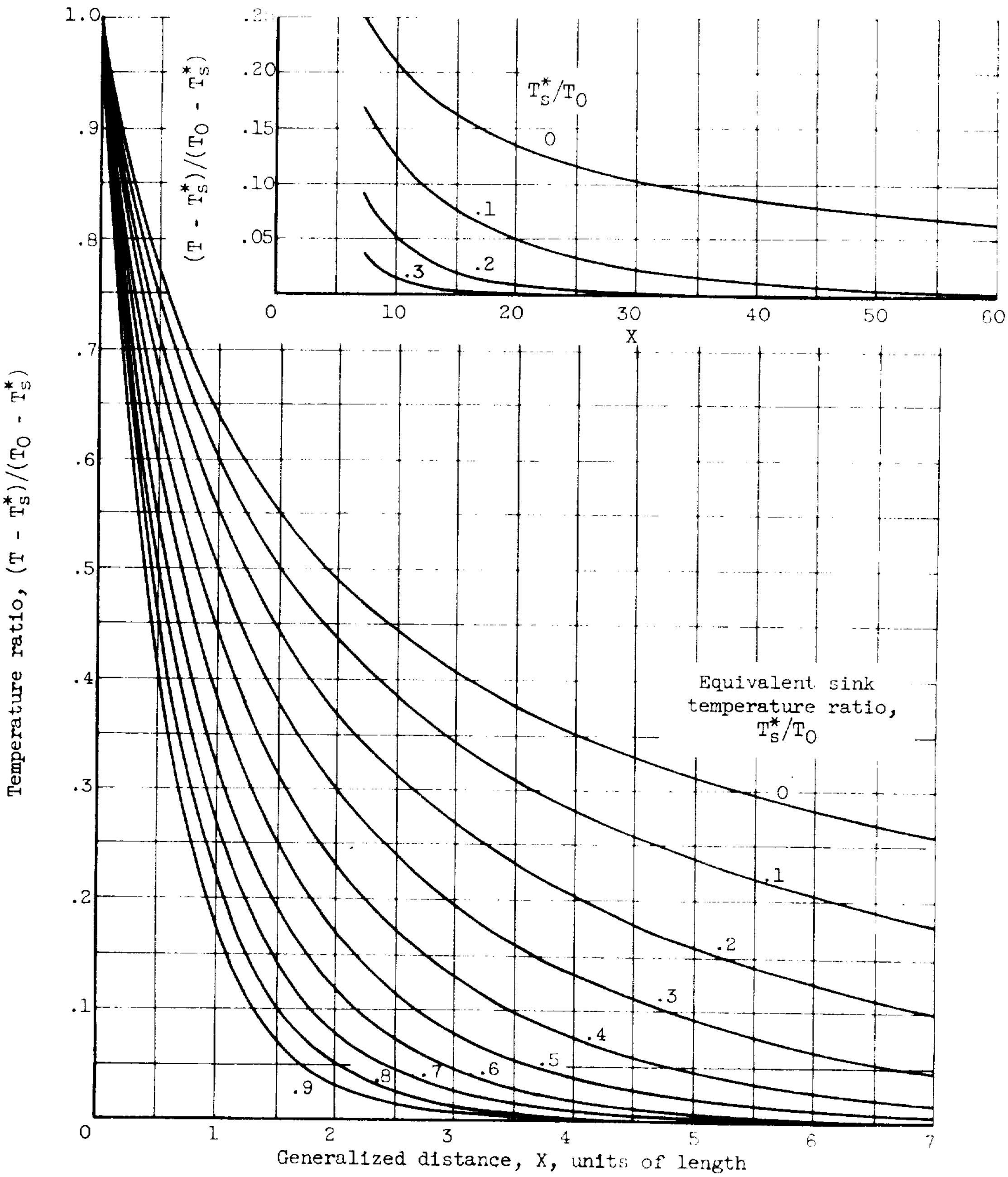
Figure 3. - Concluded. Heat-transfer characteristics of infinitely long plate at zero sink temperature.



(a) Source temperature, 1000° R.

(b) Source temperature, 2000° R.

Figure 4. - Illustrative variation of temperature ratio with distance from heat source for infinite plate.

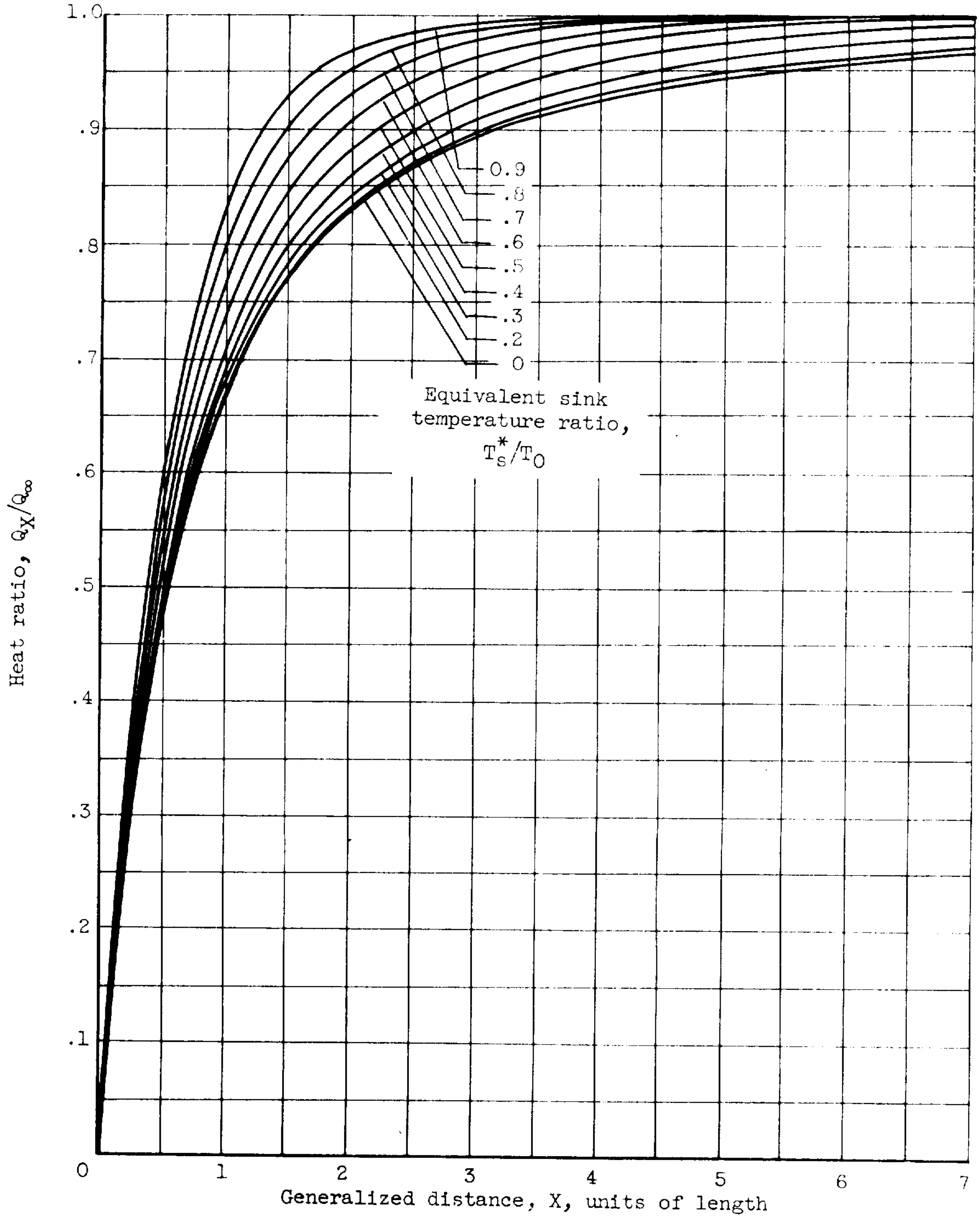


(a) Temperature profiles.

Figure 5. - Variation of heat-transfer characteristics with generalized distance parameter for flat plate of infinite length.

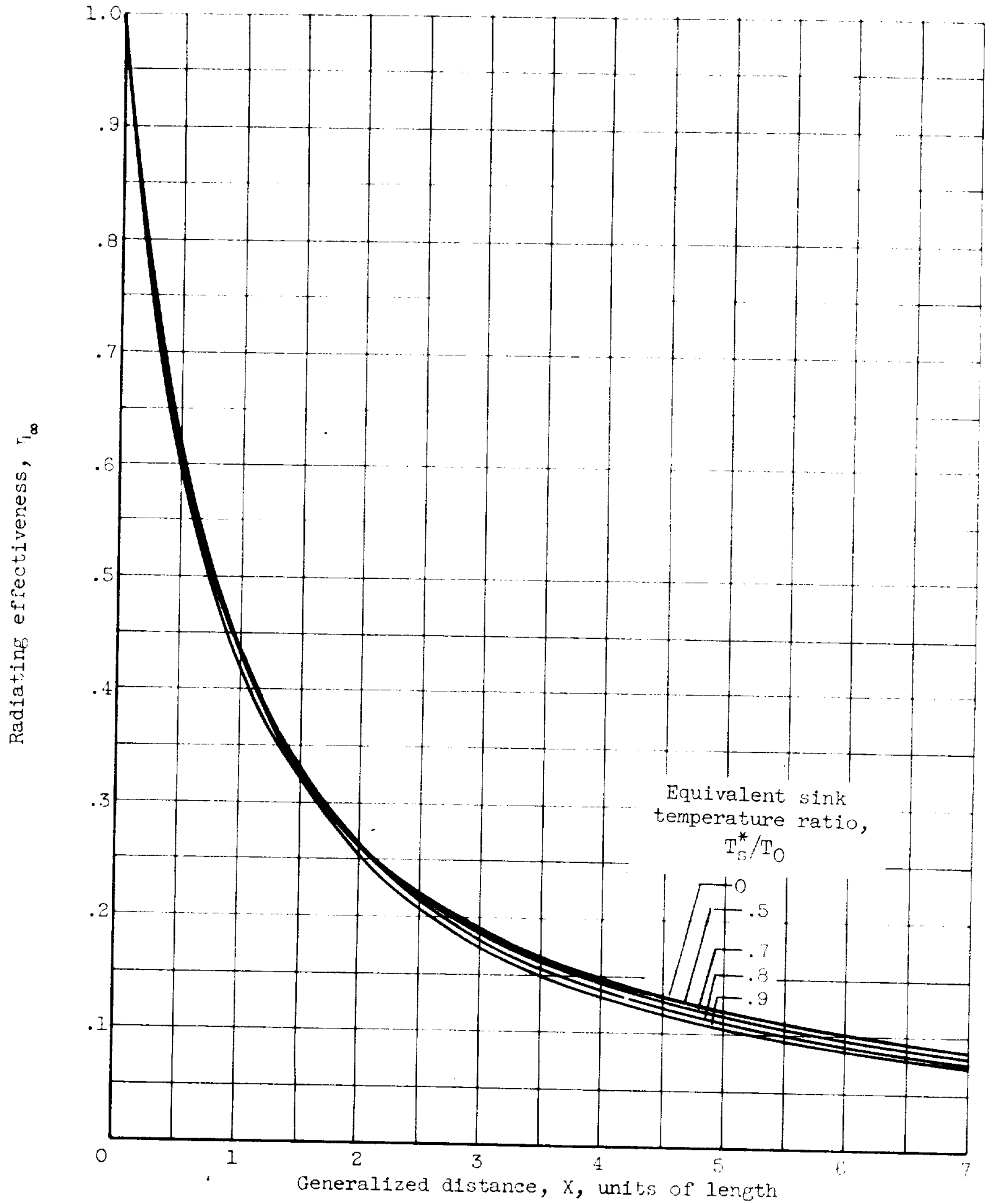
E-529

CL-6 back



(b) Heat ratio.

Figure 5. - Continued. Variation of heat-transfer characteristics with generalized distance parameter for flat plate of infinite length.



(c) Radiating effectiveness.

Figure 5. - Concluded. Variation of heat-transfer characteristics with generalized distance parameter for flat plate of infinite length.

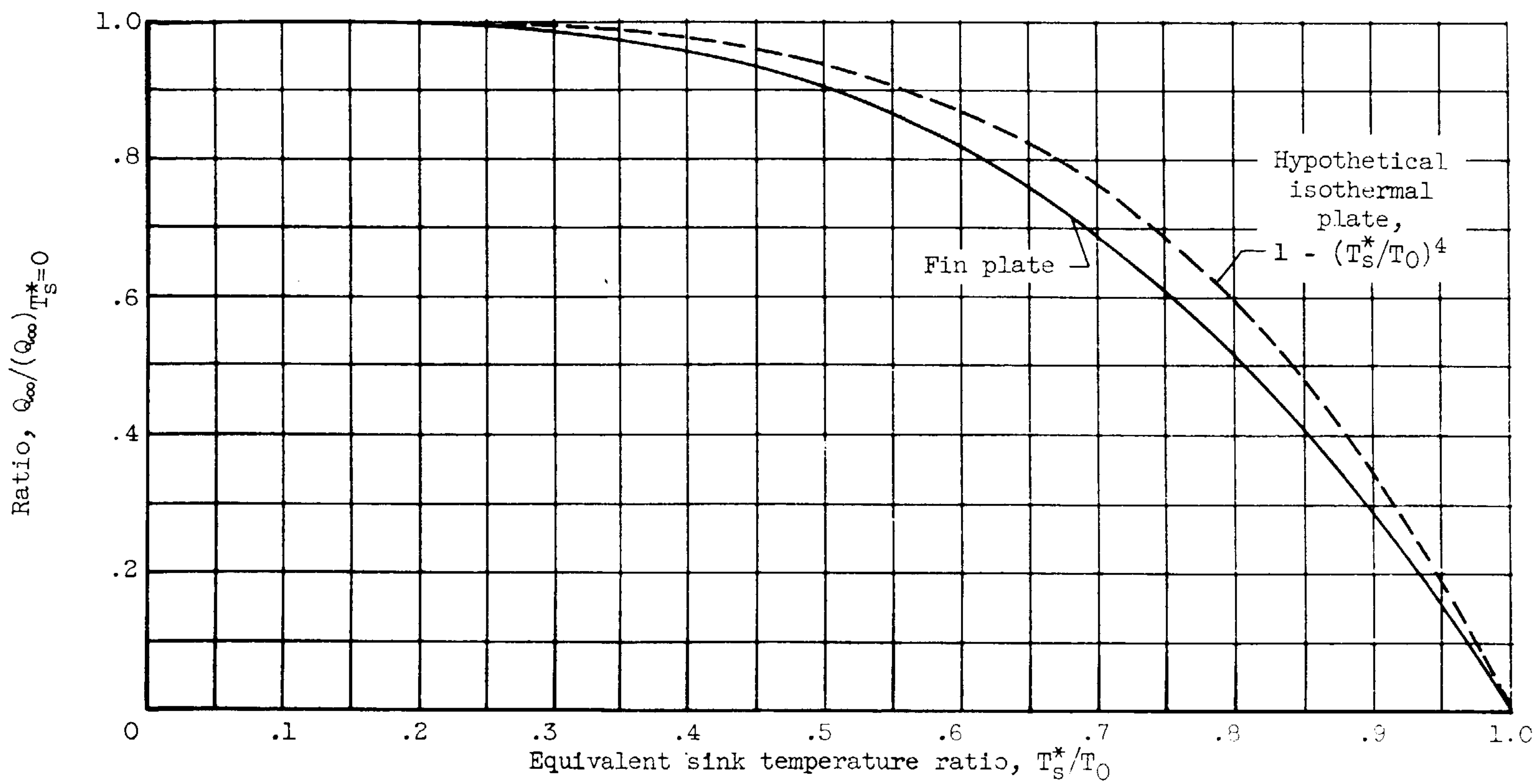


Figure 6. - Effect of sink temperature ratio on maximum heat radiated from plate of infinite length.

E-529

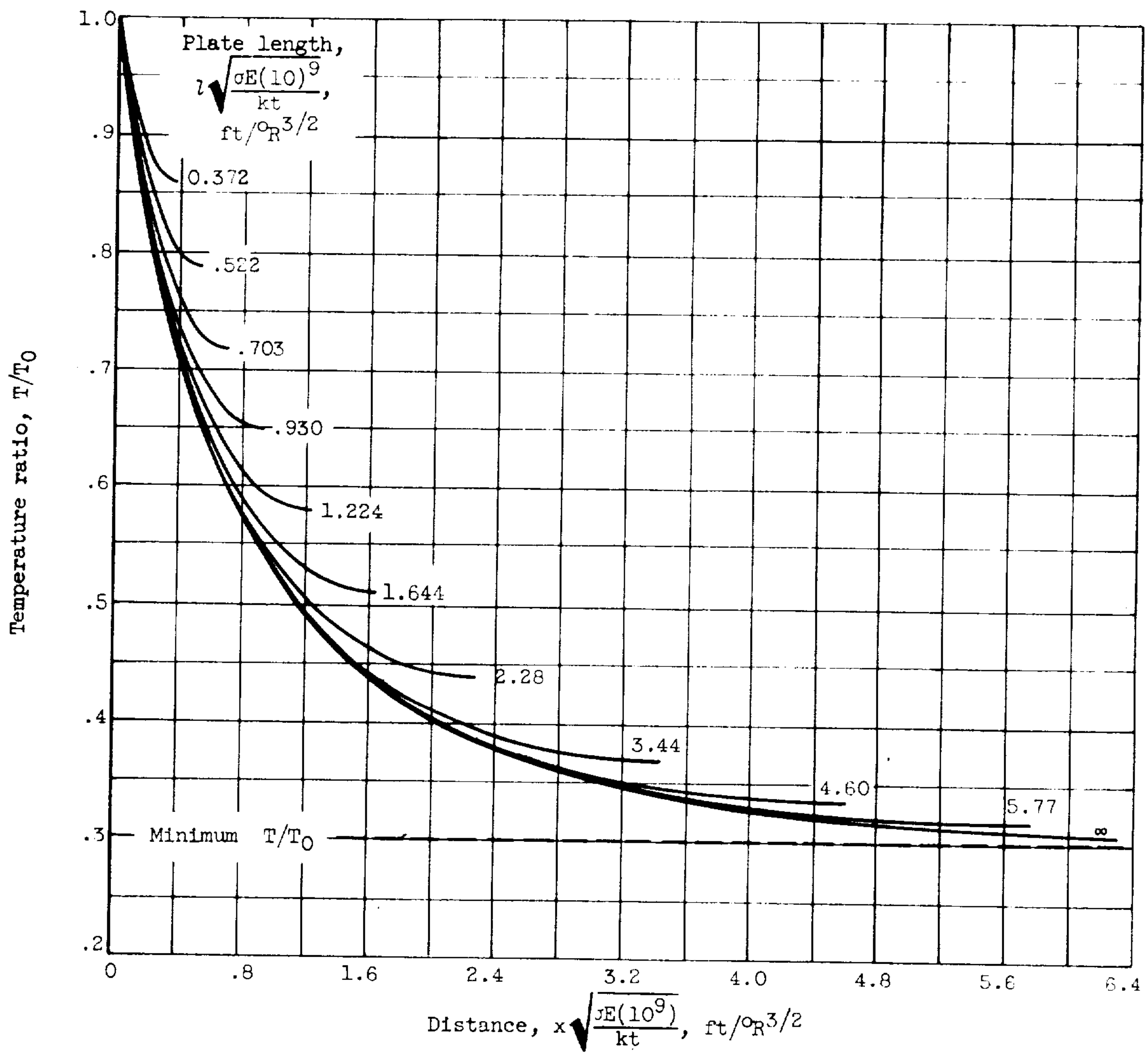


Figure 7. - Temperature-ratio profiles for various plate lengths. Source temperature, 1500° R; equivalent sink temperature, 450° R.

E-529

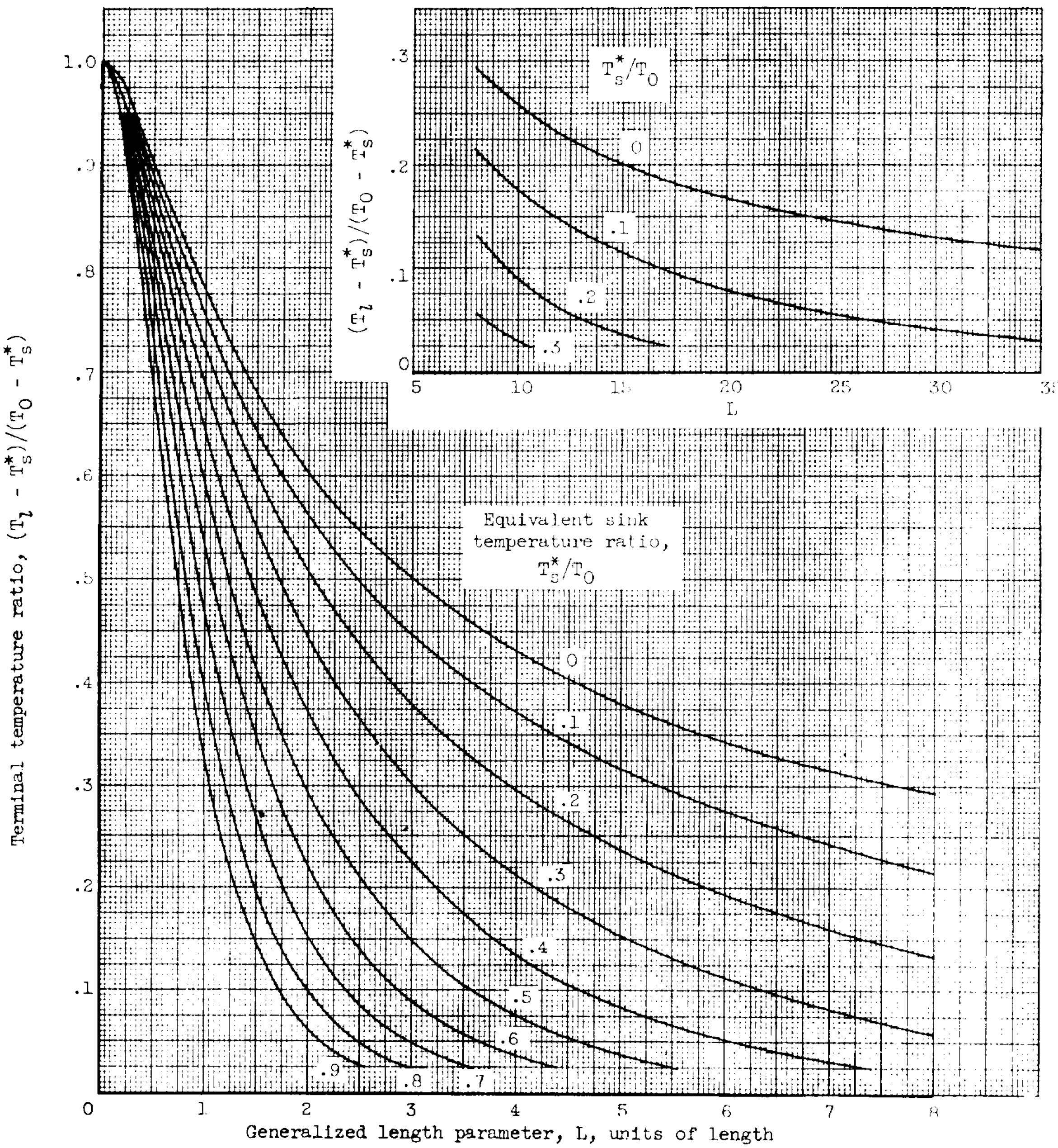


Figure 8. - Variation of terminal temperature ratio with fin plate length.

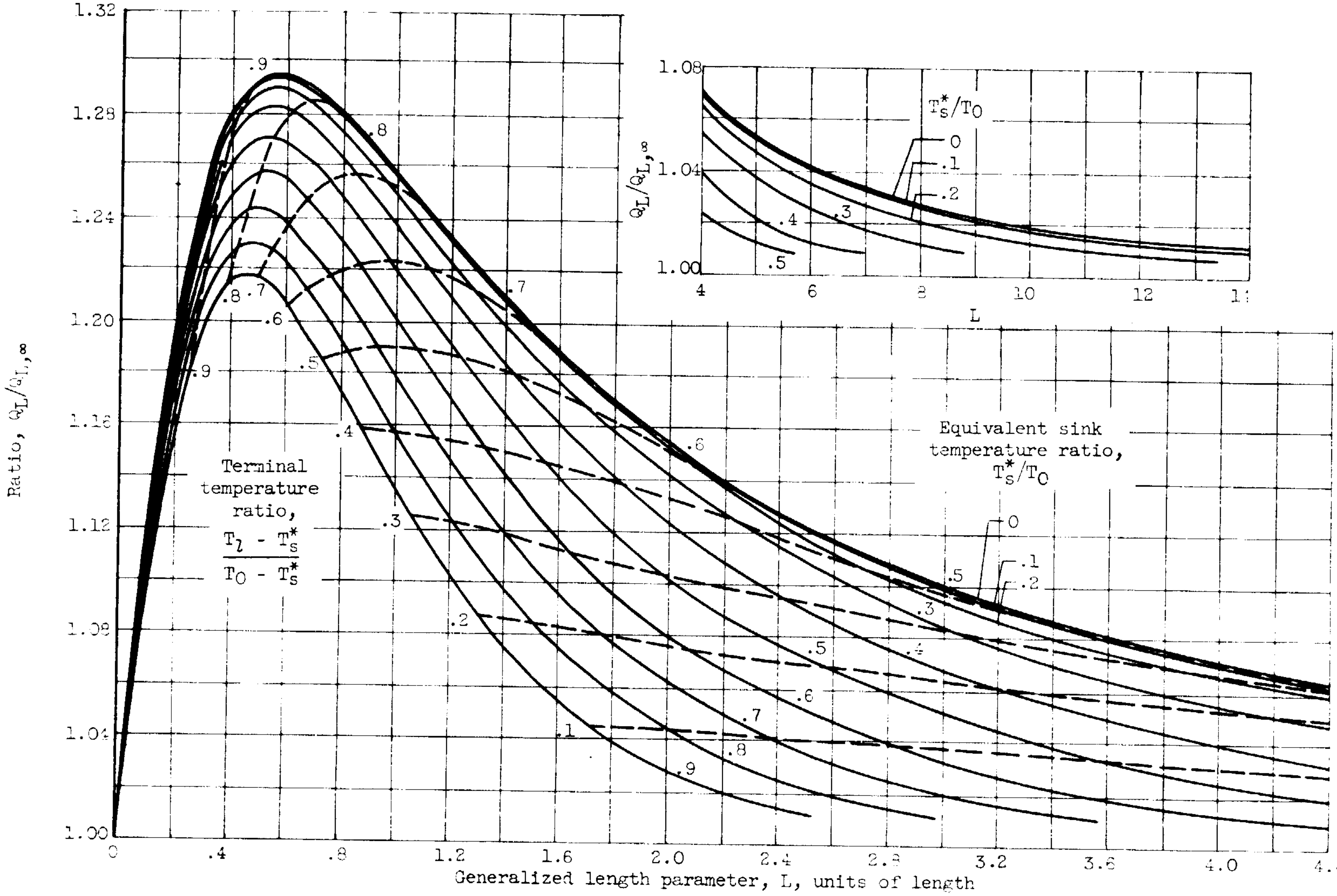


Figure 9. - Ratio of heat radiated from plate of finite length to heat radiated from infinitely long plate at distance equal to length of finite plate.

E-529

CL-7

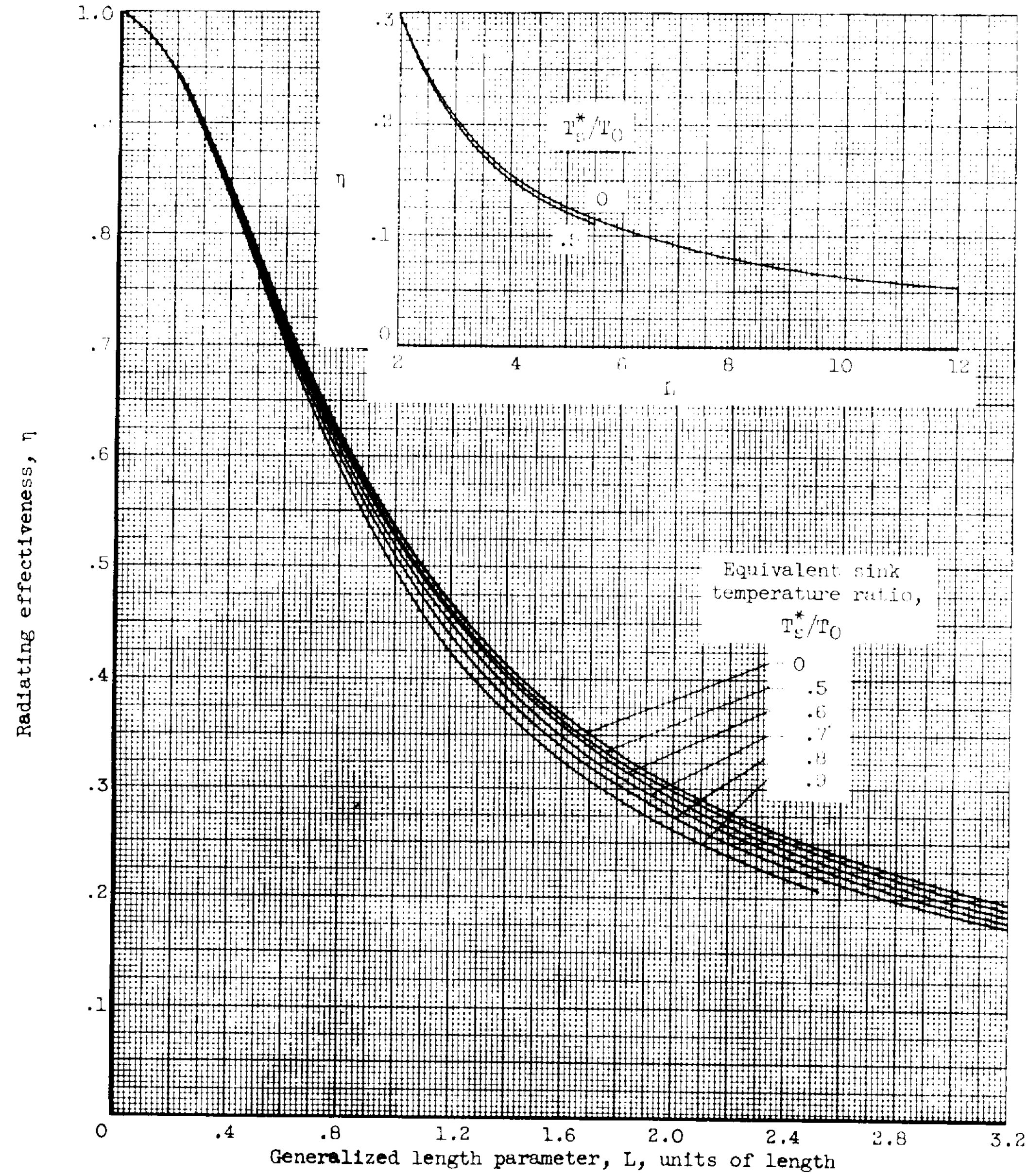


Figure 10. - Variation of radiating effectiveness with fin plate length.

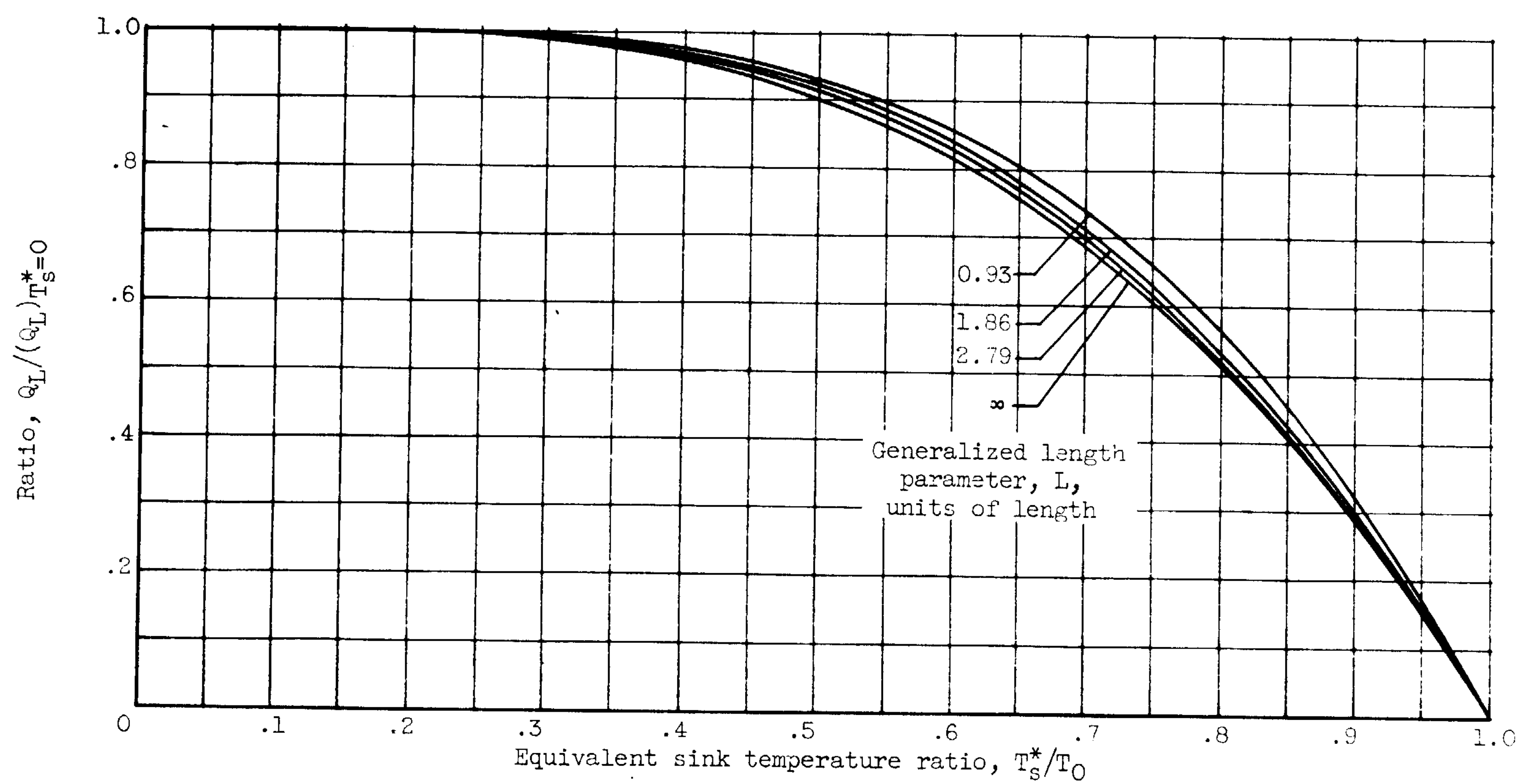


Figure 11. - Effect of equivalent sink temperature ratio on total heat radiated from plates of various lengths.

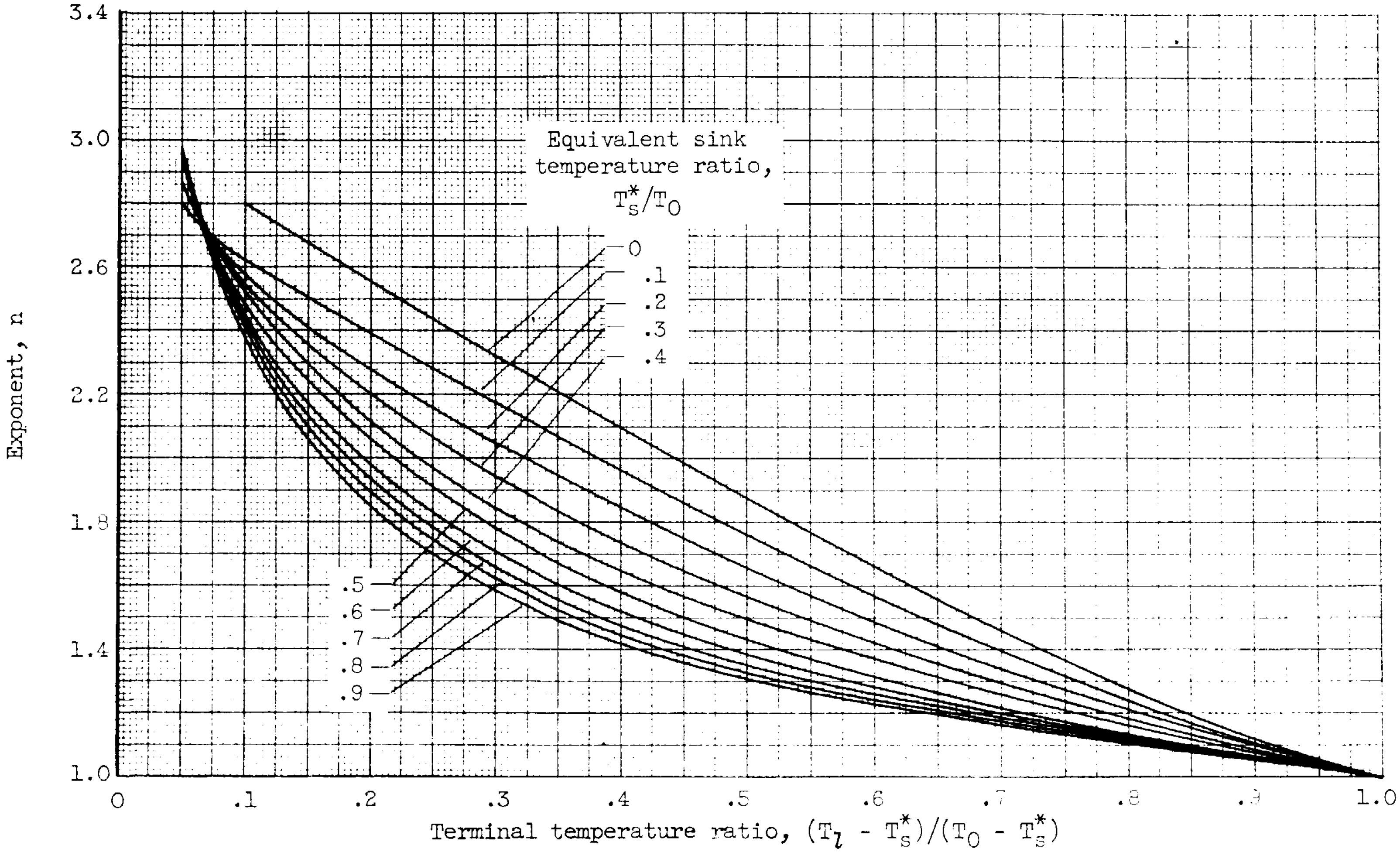


Figure 12. - Values of exponent n for calculation of variation of temperature ratio along fin plate of finite length.

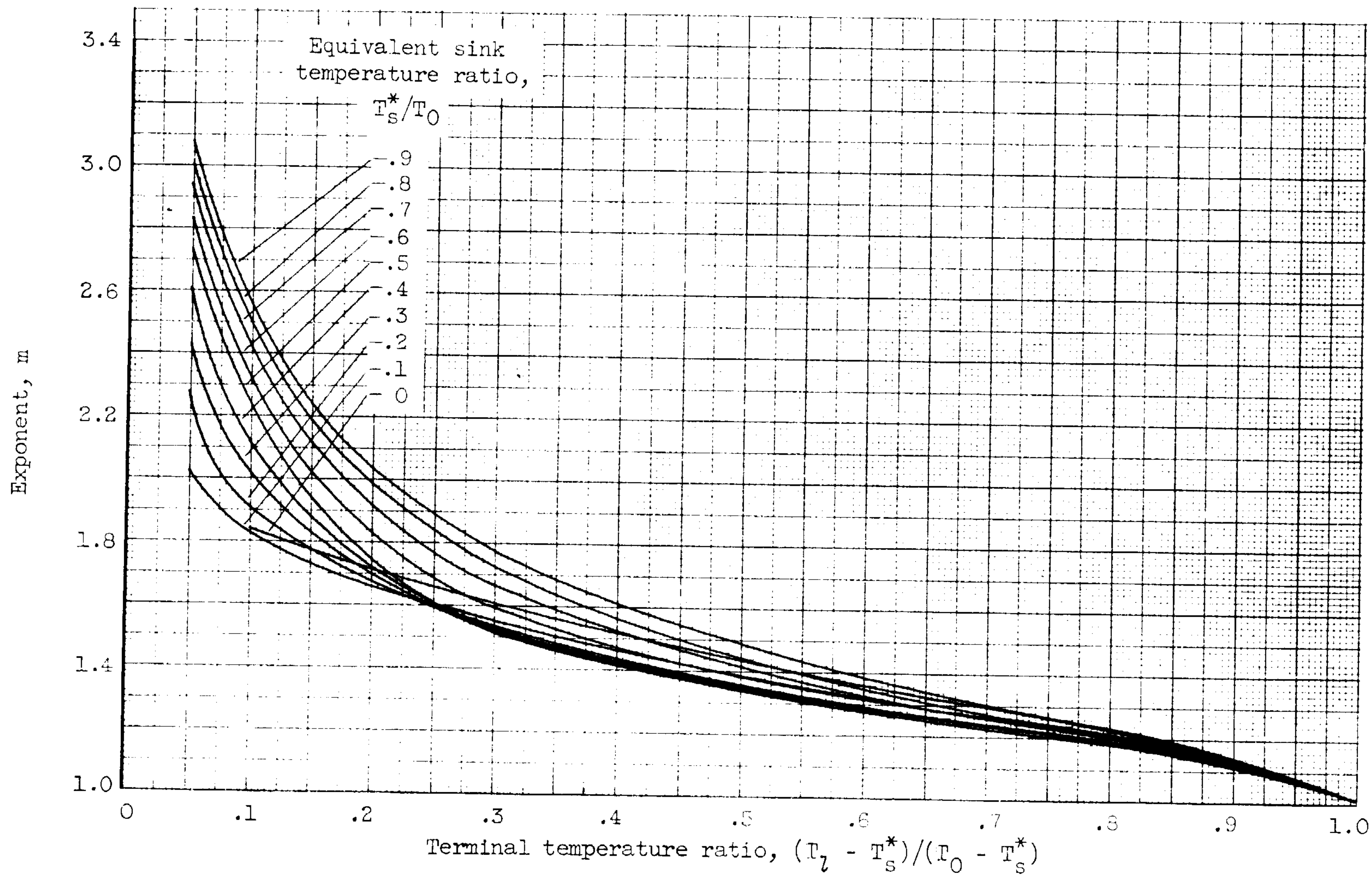
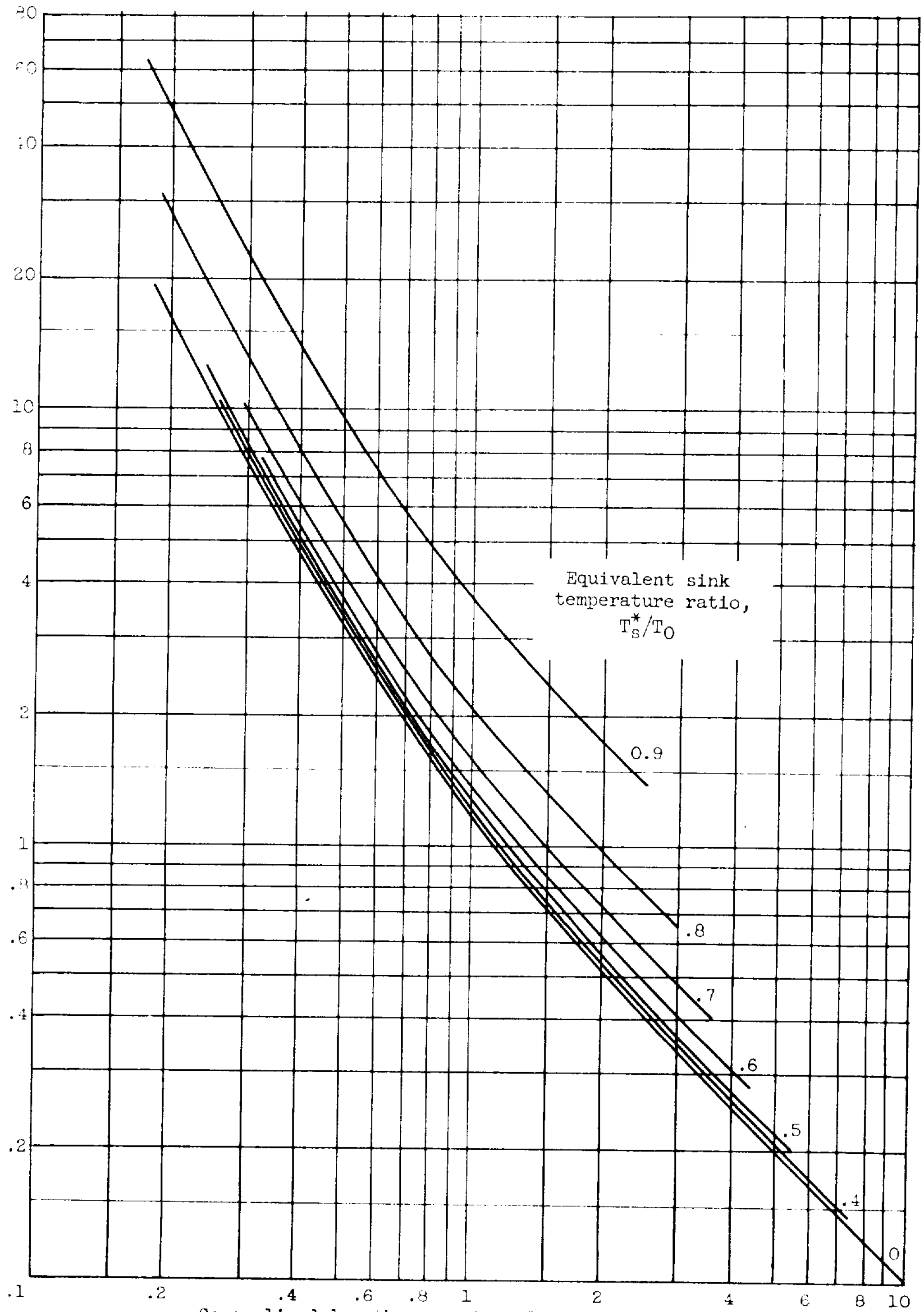


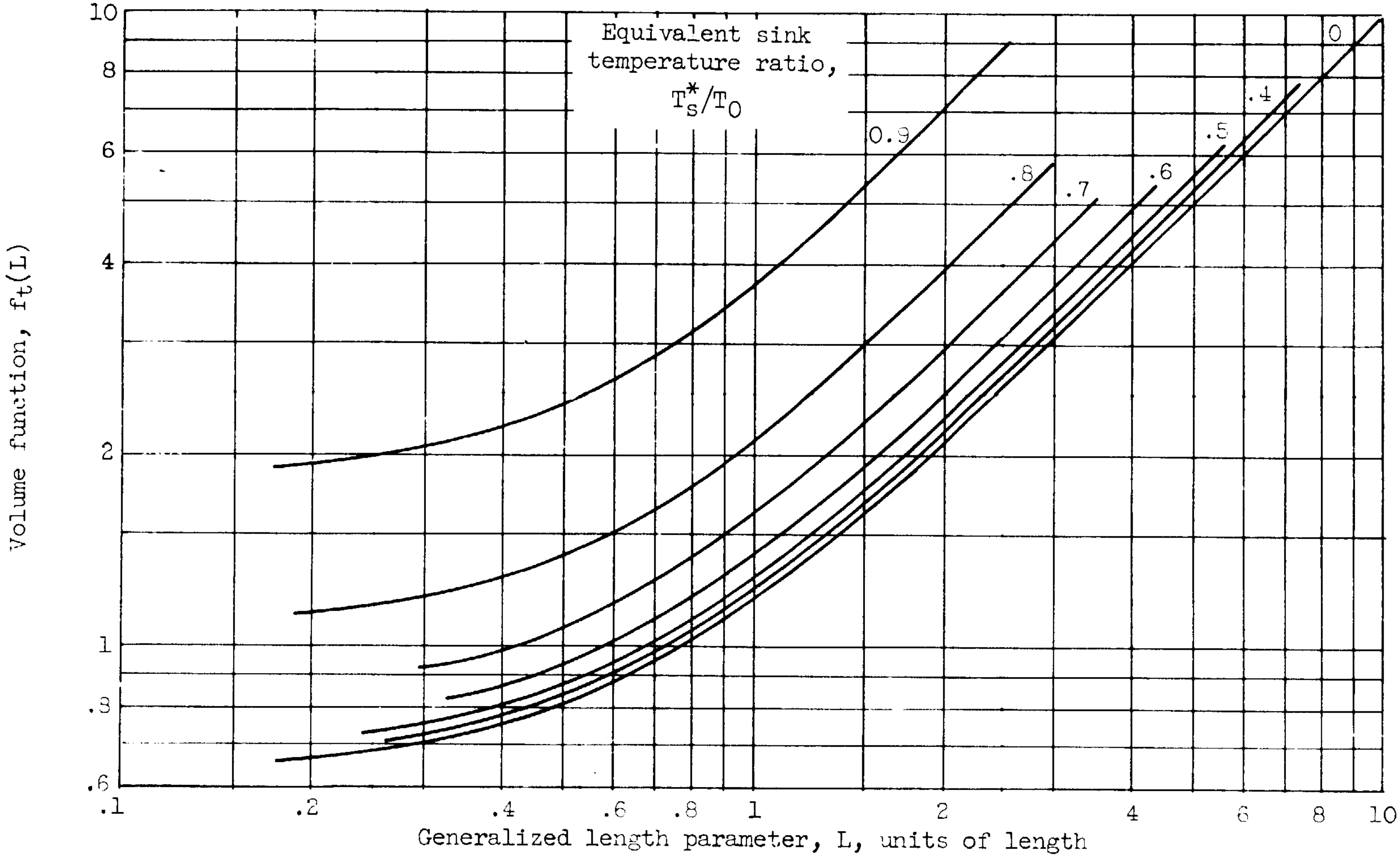
Figure 13. - Values of exponent m for calculation of variation of heat radiated along fin plate of finite length.

E-529



(a) Fixed length.

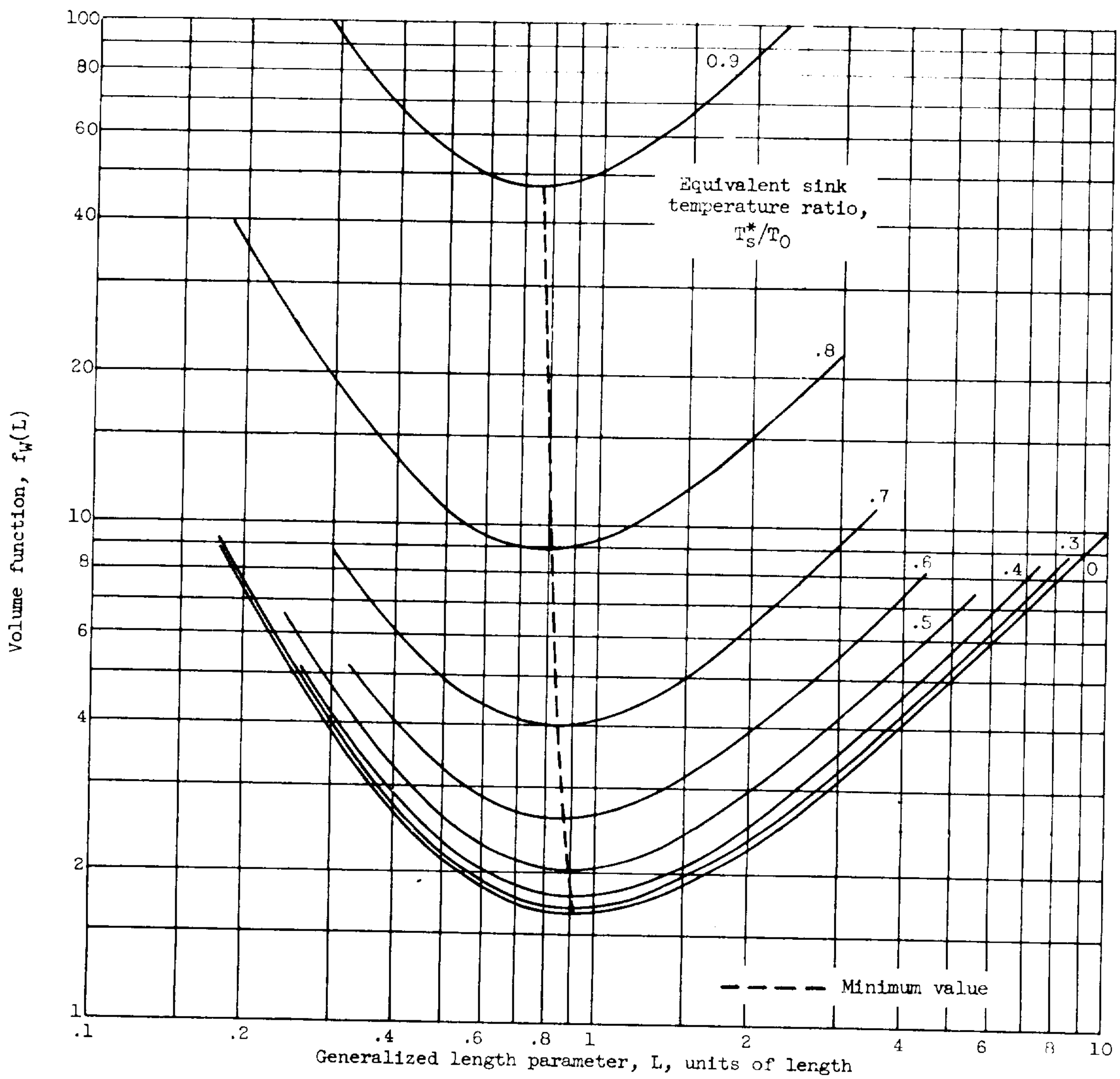
Figure 14. - Variation of volume function with generalized length parameter for determination of minimum plate volume.



(b) Fixed thickness.

Figure 14. - Continued. Variation of volume function with generalized length parameter for determination of minimum plate volume.

E-529



(c) Fixed width.

Figure 14. - Concluded. Variation of volume function with generalized length parameter for determination of minimum plate volume.

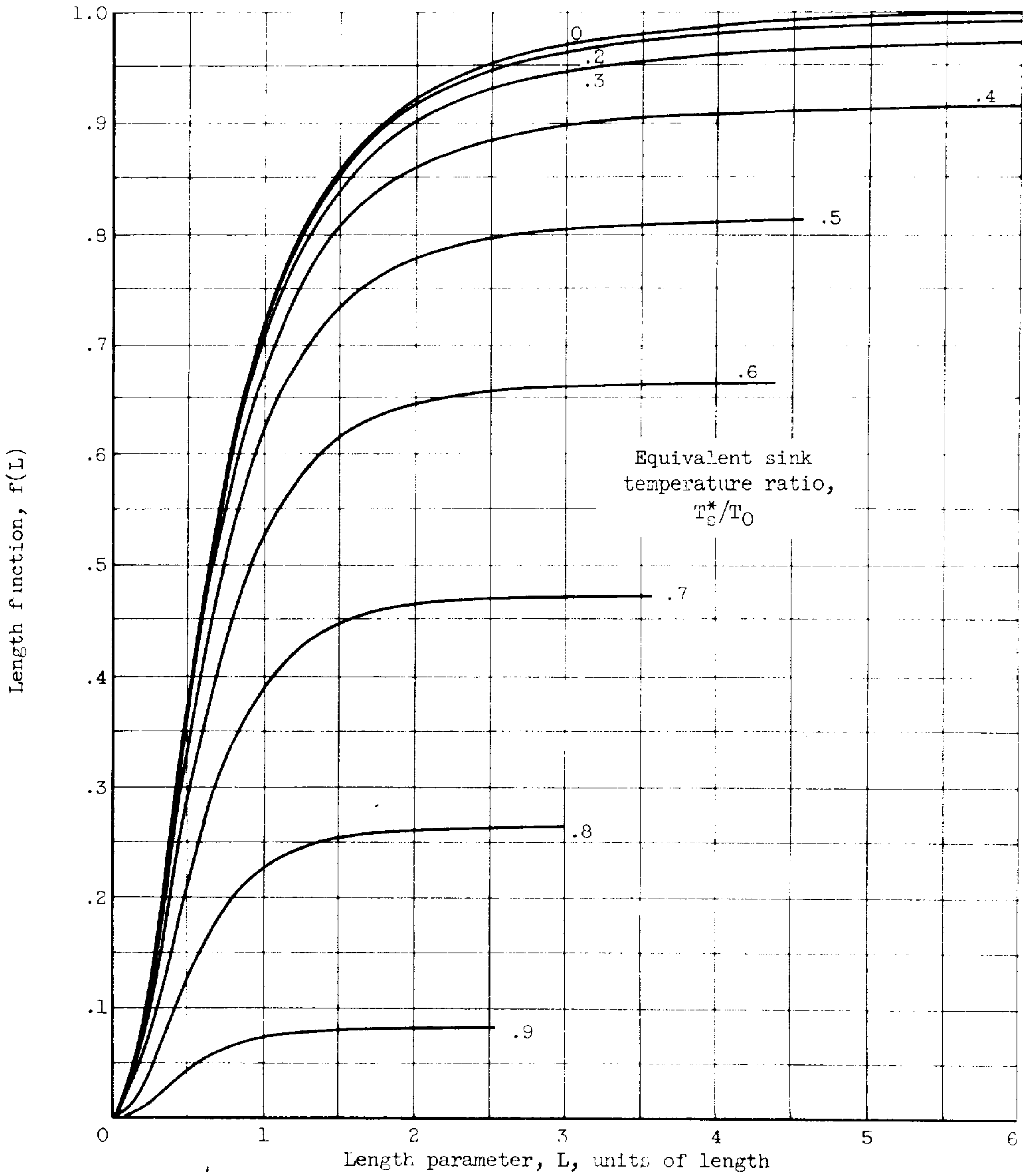


Figure 15. - Variation of length function with generalized length parameter.

E-529

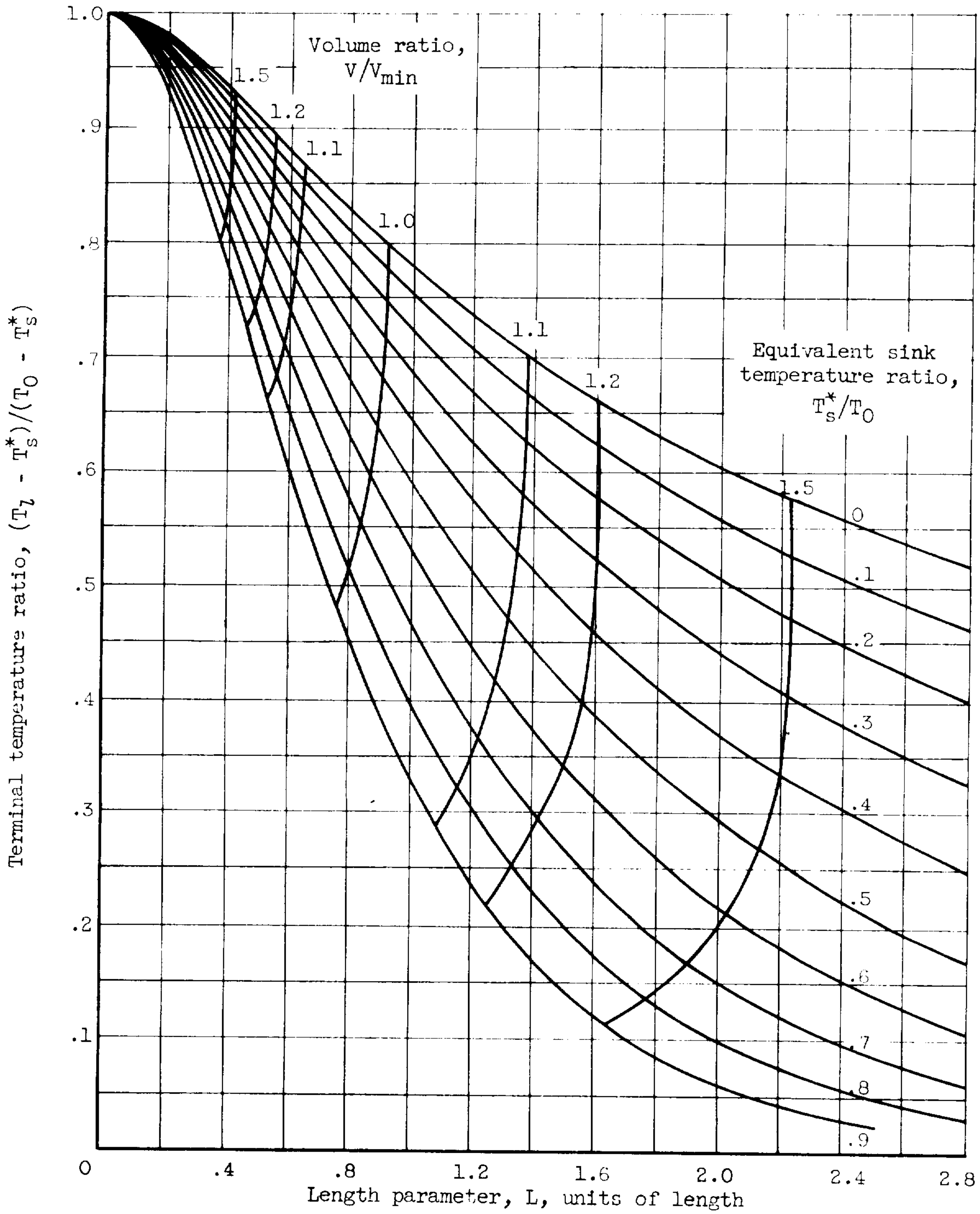
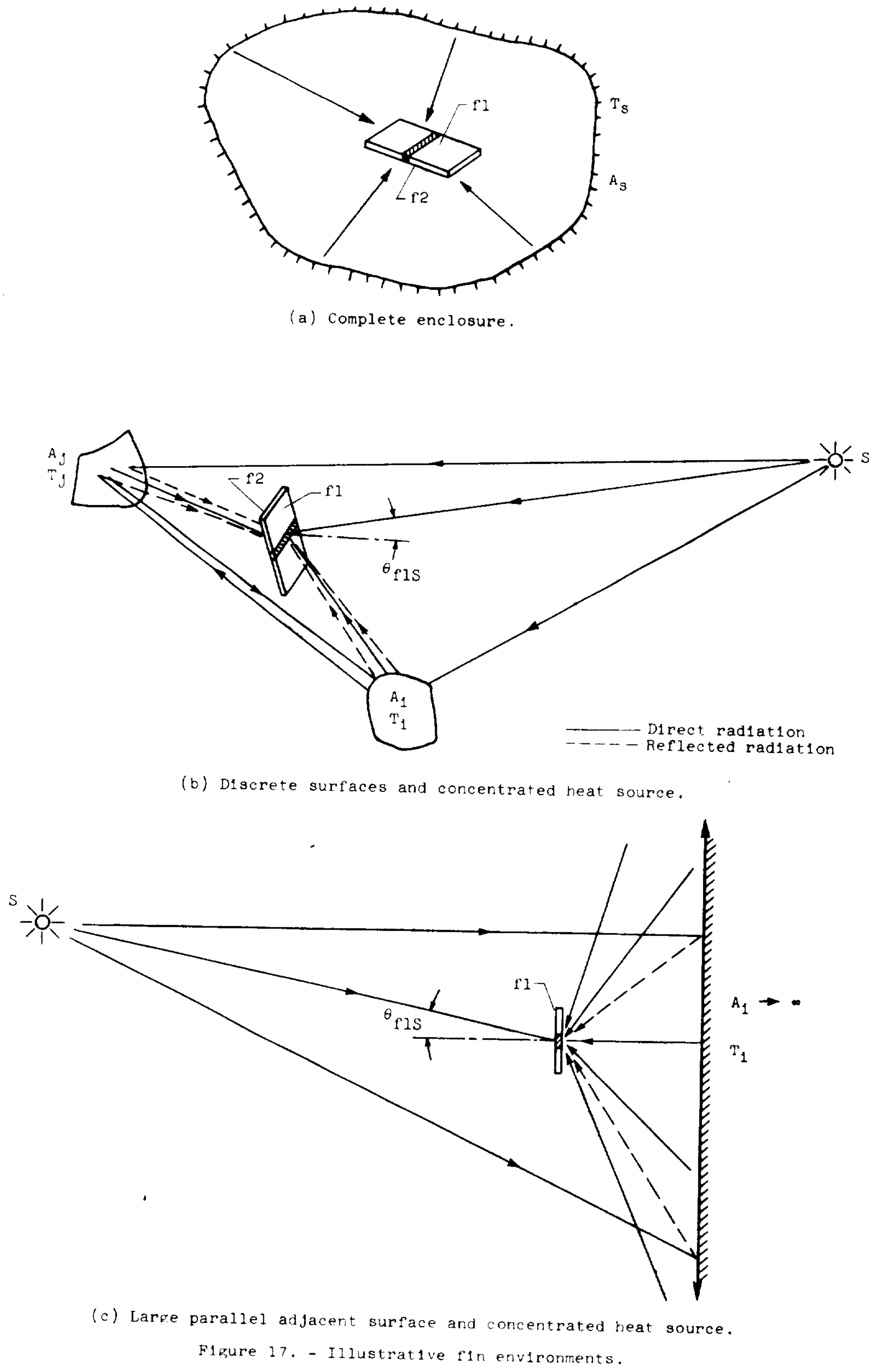


Figure 16. - Variation of terminal temperature ratio with fin plate length showing lines of constant plate volume ratio for fixed plate width.

E-529



(c) Large parallel adjacent surface and concentrated heat source.

Figure 17. - Illustrative fin environments.

E-529

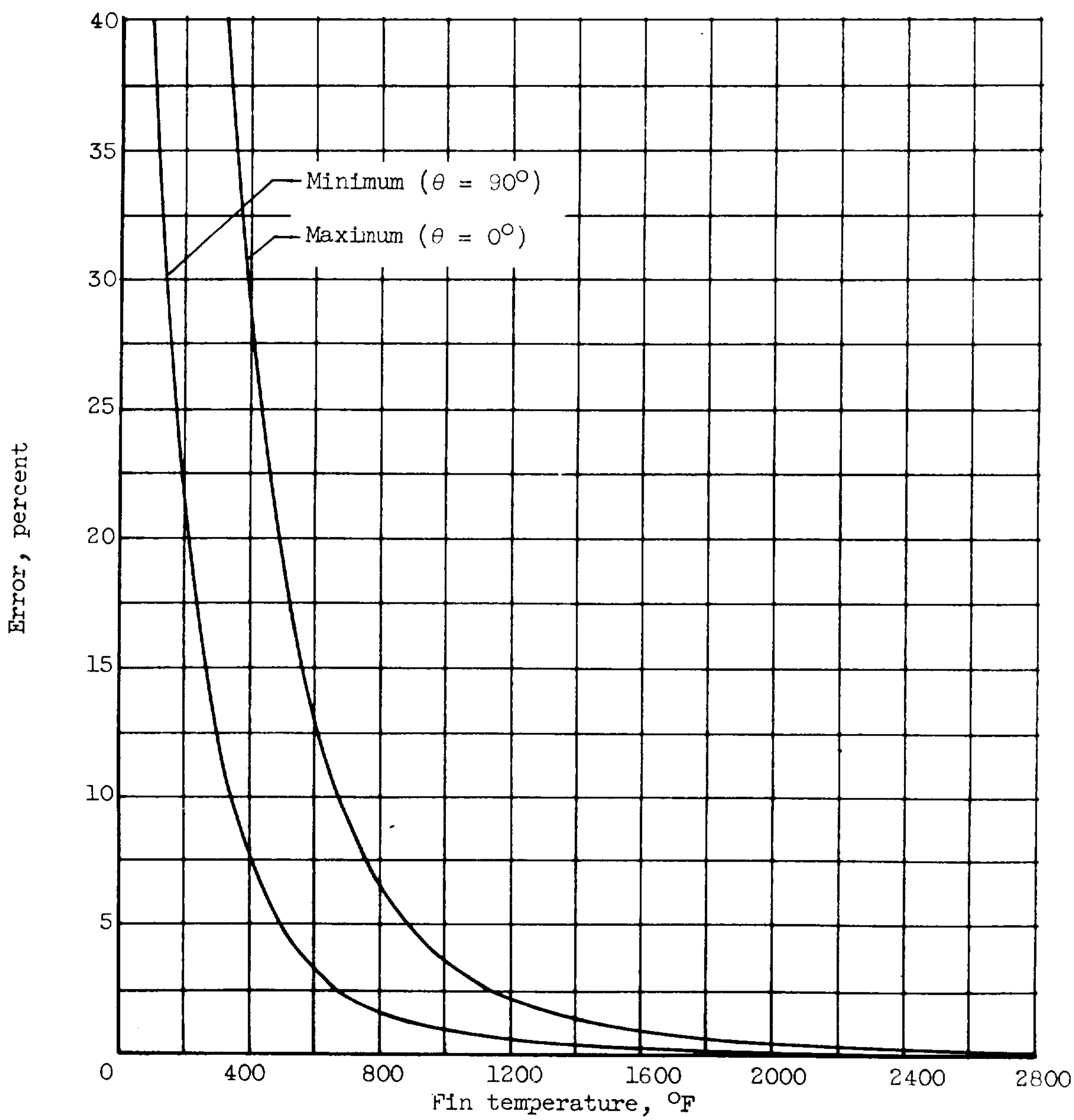


Figure 18. - Approximate error involved in neglecting solar radiation and earth sink temperature in radiation from plate orbiting parallel to earth's surface. Earth temperature, 80° F; solar heat flux, 429 Btu/hr-ft^2; absorptivity = emissivity.

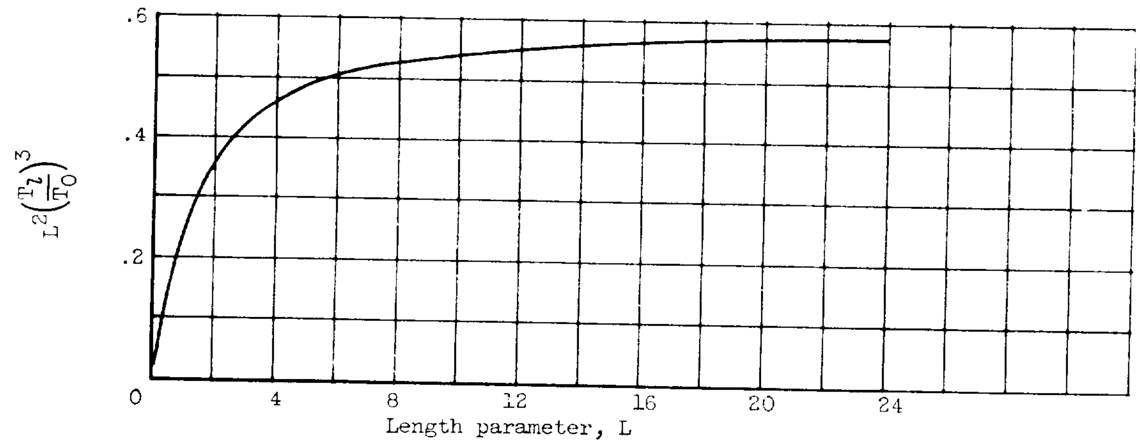


Figure 19. - Variation of function with generalized length parameter.

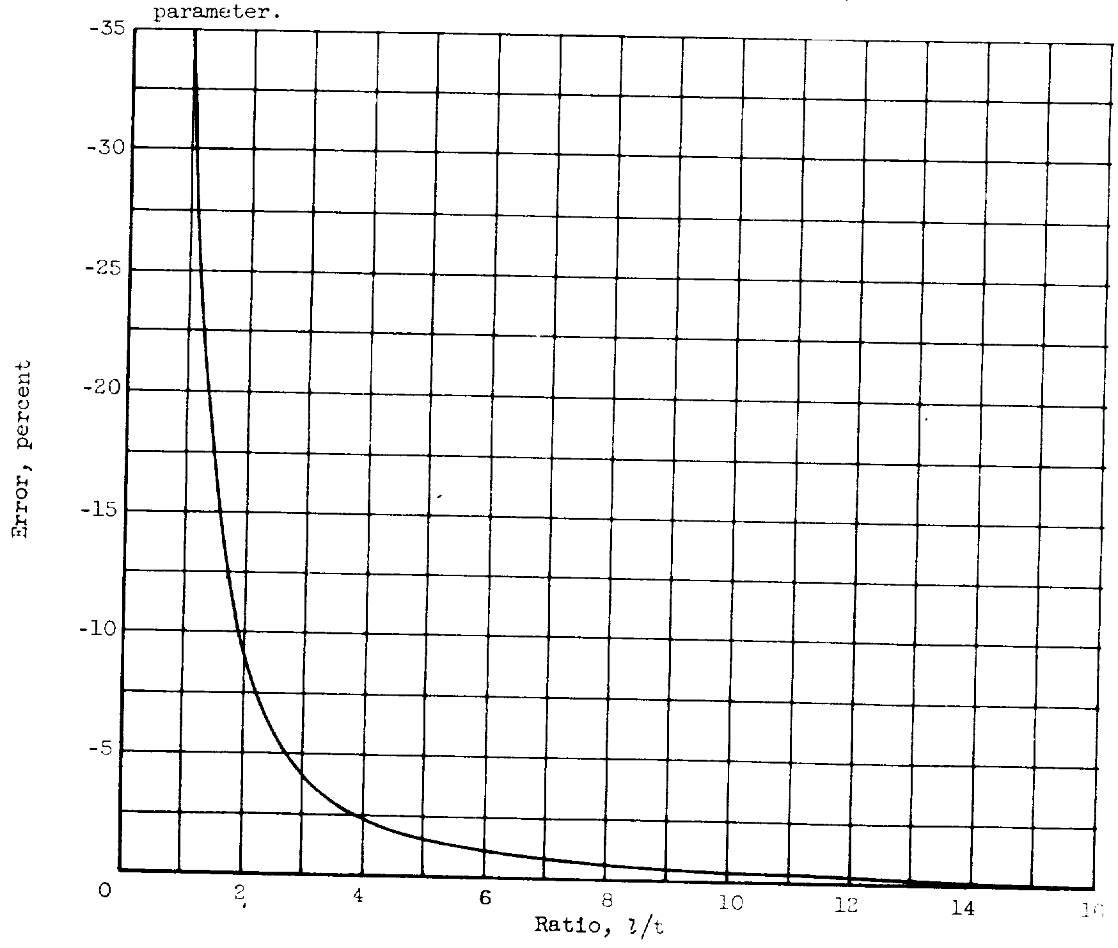


Figure 20. - Variation of maximum error in constant of integration due to neglect of radiant heat loss from exposed end edge.